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Semigroups of density-continuous functions

Introduction. The inspiration for our work comes from the following results concerning the semigroup of continuous selfmaps of a topological space and, in the case of the space being the real line, its subsemigroup of differentiable selfmaps.

Theorem. (Gavrilov [6], Magill [9], Malcev [12], Shneperman [19]) *If X is a completely regular topological space which contains a simple arc then the semigroup $S(X)$ of all continuous selfmaps on X has the inner automorphism property.*

Theorem. (Magill [11]). *The semigroup of differentiable real-valued functions of a real variable has the inner automorphism property.*

A semigroup S of selfmaps of a set X is said to have the *inner automorphism property* if every automorphism Φ of S is of the form $\Phi(f) = h \circ f \circ h^{-1}$ with $h \in S$ being a bijection of X.

In this paper we consider the semigroup of selfmaps of $\mathbf{R}$ which are continuous with respect to the density topology, that is, maps $f:\mathbf{R} \to \mathbf{R}$ where the topology on both the domain and the range is the density topology (see [7], or [17] p. 90, or [20] for the definition of the density topology). Such maps are called *density-continuous* functions. It is well known that the density topology is completely regular (see [8], [20], and [21]). However, countable sets never have cluster points in the density topology while bounded infinite countable sets do have cluster points in the natural topology. This implies that the only continuous mappings from $\mathbf{R}$ equipped with the natural topology into $\mathbf{R}$ equipped with the density topology are constant maps. Thus the real line equipped with the density topology contains no simple arcs. Consequently, the semigroup of density-continuous selfmaps of $\mathbf{R}$ cannot be shown to have the inner automorphism property by using the first of the above theorems. Nonetheless, we shall prove that this semigroup does have the inner automorphism property (see Theorem 3.6 below).

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Concerning the above theorems, we shall consider the subsemigroup of differentiable density-continuous selfmaps of $\mathbf{R}$ and the two subsets consisting of those selfmaps which are approximately differentiable and those selfmaps which are almost everywhere approximately differentiable, respectively. In Section 1, we shall prove that these two subsets are indeed subsemigroups. Moreover, in Section 3, we shall prove that all three subsemigroups also have the inner automorphism property (Theorems 3.6 and 3.9).

Section 2 includes a discussion of properties of density-continuous functions and a lemma needed in the proof of the main theorems given in section 3. In section 4, we discuss a possible alternate proof that the semigroup of density-continuous selfmaps of $\mathbf{R}$ has the inner automorphism property.

The following standard notation will be used:

R – the set of all real numbers;

N – the set of all natural numbers;

$\overline{d}(A,x), \underline{d}(A,x)$, and $d(A,x)$ – the upper, lower, and ordinary (respectively) densities of a set $A \subset \mathbf{R}$ at a point $x \in \mathbf{R}$;

$|E|$ – the outer Lebesgue measure of a set $E \subset \mathbf{R}$.

All semigroups considered are of selfmaps of $\mathbf{R}$ with composition as the operation.

1. Subsemigroups.

It is obvious that the class of differentiable density-continuous functions is a semigroup. We also have the following:

1.1. Theorem. *If $f, g: \mathrm{R} \to \mathbf{R}$ are approximately differentiable and g is density-continuous then $f \circ g$ is approximately differentiable.*

Proof. See [15], Theorem 5.2.

1.2. Corollary. *The class of approximately differentiable density-continuous functions $f: \mathrm{R} \to \mathrm{R}$ is a semigroup.*

1.3. Theorem. *If $f, g: \mathbf{R} \to \mathbf{R}$ are density-continuous and approximately differentiable almost everywhere then so is $f \circ g$.*

Proof. Let Z_f be the set where f is not approximately differentiable. It is enough to show that $f \circ g$ is approximately differentiable almost everywhere on $g^{-1}(Z_f)$.

The class of all level sets $g^{-1}(\{z\})$ for $z \in Z_f$ which are of positive measure is clearly at most countable, and f is constant on each such level set, therefore $f \circ g$ is approximately differentiable almost everywhere on the union of the entire class.

The remaining part of $g^{-1}(Z_f)$ is made out of level sets of measure zero. Since g is density-continuous, any point $x \in g^{-1}(z)$ lying in such a level set must be a dispersion point of $g^{-1}(Z_f) \setminus g^{-1}(\{z\})$, as Z_f has only dispersion points (see the discussion of density-continuity at a point in [15] and [16]). Then x is also a dispersion point of $g^{-1}(Z_f)$, and the set of those dispersion points is of measure zero, so the remaining part of $g^{-1}(Z_f)$ is of measure zero.

This shows that $f \circ g$ is approximately differentiable almost everywhere on $g^{-1}(Z_f)$. The proof is ended.

1.4. Corollary. *The class of density-continuous functions which are approximately differentiable almost everywhere is a semigroup.*

1.5. Notation. We have just finished introducing the semigroups which we will investigate. Here is the entire list, with the notation used:

$\mathcal{C}_\mathcal{D}$ – density-continuous functions.

$d\mathcal{C}_\mathcal{D}$ – differentiable density-continuous functions.

$a\mathcal{C}_\mathcal{D}$ – approximately differentiable density-continuous functions.

$z\mathcal{C}_\mathcal{D}$ – almost everywhere approximately differentiable density-continuous functions.

We have the following obvious inclusions:

$$d\mathcal{C}_\mathcal{D} \subset a\mathcal{C}_\mathcal{D} \subset z\mathcal{C}_\mathcal{D} \subset \mathcal{C}_\mathcal{D}. \tag{1}$$

All of them are proper. We will discuss that later.

2. Properties of density-continuous functions.

2.1. While this work was being written the following facts concerning density-continuous functions were discovered:

Theorem. *(a) Real-analytic functions are density continuous.*

(b) The class of density-continuous functions is not additive. In fact, there exists a density-continuous f such that $g(x) = x + f(x)$ is not density-continuous.

(c) There exists a C^∞ function which is not density-continuous.

Proof. All of the above are shown in [4]. Also, the first two statements are proved in [3].

2.2. Lemma. *Let $a < b \leq c < d$. Then there exists an everywhere differentiable, density-continuous function $f: \mathrm{R} \to \mathrm{R}$ such that:*
(i) $f(x) = 0$ if $x \notin (a,d)$; and $f(x) = 1$ if $b \leq x \leq c$.
(ii) $f'(x) > 0$ if $x \in (a,b)$; and $f'(x) < 0$ if $x \in (c,d)$.

Proof. The above follows easily from the fact that polynomials are density-continuous.

2.3. Examples. Let

$$x_n = \sum_{i=n}^{+\infty} \frac{1}{i^2},\ y_n = \sum_{i=n}^{\infty} \frac{1}{2^i} \tag{2}$$

for $n \in \mathrm{N}$. Define $g: [0, x_1] \to [0,1]$ as follows: $g(x_n) = y_n$ for every $n \in \mathrm{N}$, $g(0) = 0$, and g is linear in every interval $[x_{n+1}, x_n]$ for $n \in \mathrm{N}$. In [13], Theorem 3, it is shown that, for $f = g^{-1}$, f is a homeomorphism of $[0,1]$ onto $[0, x_1]$ which preserves density points, while g does not preserve density points. This implies (see [15]) that g is density-continuous, while g^{-1} is not. One can also easily see that

$$\lim_{\delta \to 0+} \frac{g(\delta)}{\delta} = \lim_{n \to \infty} \frac{y_n}{x_n} = 0, \tag{3}$$

so that the right-hand derivative of g at zero exists and equals zero. By modifying g, to be differentiable, in a sequence of intervals centered at the points x_n, $n \in \mathrm{N}$, such that the union of those intervals has a dispersion point at zero, one can construct an increasing density-continuous function $h: [0, x_1] \to [0,1]$, where $x_1 > 0$, such that h is a bijection, is differentiable, $h'(0) = 0$, and h^{-1} is not density-continuous.

As stated in [14], there are continuous functions which are not density-continuous. For example, if $\bigcup_{n=1}^{+\infty} [a_n, b_n]$ is an interval set at zero with upper outer density positive at zero, and

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x \in [a_n, b_n] \text{ for some } n \in \mathrm{N}, \\ 0 & \text{for x} = 0, \\ \text{linear} & \text{in the intervals } [b_{n+1}, a_n] \text{ for every } n \in \mathrm{N}, \end{cases} \tag{4}$$

then f is continuous and not density-continuous at zero. Since every piecewise-linear function is clearly density-continuous, and f is a uniform limit of a sequence of such functions (as is every continuous function), we can have a uniformly convergent sequence of density-continuous functions whose limit is not density-continuous.

2.4. Theorem. *The inclusions*

$$d\mathcal{C}_{\mathcal{D}} \subset a\mathcal{C}_{\mathcal{D}} \subset z\mathcal{C}_{\mathcal{D}} \subset \mathcal{C}_{\mathcal{D}}. \tag{5}$$

are proper.

Proof. One can easily see that $f(x) = |x|$ belongs to $z\mathcal{C}_{\mathcal{D}}$ but not to $a\mathcal{C}_{\mathcal{D}}$. The following is an example of a function which is in $a\mathcal{C}_{\mathcal{D}}$ but not in $d\mathcal{C}_{\mathcal{D}}$.

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences of real numbers such that, for every $n \in \mathrm{N}$, $0 < a_n < b_n < c_n < a_{n+1}$ and $\bigcup_{n\in\mathrm{N}}[a_n, b_n]$ has right-hand density zero at zero. For every $n \in \mathrm{N}$ let $f_n:\mathbf{R} \to \mathbf{R}$ be an everywhere differentiable density-continuous function given by Lemma 2.1 such that

(i) $f_n(x) = 0$ if $x \notin (a_n, c_n)$; $f_n(b_n) = 1$;

(ii) $f_n'(x) > 0$ if $a_n < x < b_n$; $f_n'(x) < 0$ if $b_n < x < c_n$;

Define

$$f(x) = \sum_{n=1}^{+\infty} f_n(x). \tag{6}$$

Then $f \in a\mathcal{C}_{\mathcal{D}} \setminus d\mathcal{C}_{\mathcal{D}}$.

Finally, Krzysztof Ciesielski, Lee Larson, and Krzysztof Ostaszewski in [5] have constructed a continuous, density-continuous function which is nowhere approximately differentiable. This concludes the proof.

2.5. Remark. Recall (see [8]) that a function $f:\mathbf{R} \to \mathbf{R}$ is approximately continuous if and only it is continuous as a mapping from $\mathbf{R}$ equipped with the density topology to $\mathbf{R}$ equipped with the natural topology. Thus any density-continuous function is necessarily approximately continuous. Also, any approximately continuous bijection of $\mathbf{R}$ is actually a homeomorphism with respect to the natural topology (see [2]).

Finally, as observed in [15], density-continuous bijections of $\mathbf{R}$ are precisely those homeomorphisms h for which h^{-1} preserves density points (homeomorphisms preserving density points are discussed in [1] and [13]).

3. Proofs of main theorems.

We will turn now to the question which interests us most in this work - do the semigroups considered have the inner automorphism property? We will start by recalling a classical lemma due to J. Schreier (see [18]).

3.1. Lemma. *Let X be a set, $S(X)$ a semigroup of selfmaps of X such that every constant mapping is in $S(X)$, and let Φ be an automorphism of $S(X)$. Then there exists a bijection $h: X \to X$ such that $\Phi(f) = h \circ f \circ h^{-1}$ for every $f \in S(X)$.*

3.2. Definition. Obviously, since $S(X)$ contains every constant function, the bijection h is unique.Let us, from now on, denote by S any of the four semigroups discussed in section 2. The Lemma 3.1 is applicable and each automorphism Φ of S is determined by a unique bijection $h: \mathbf{R} \to \mathbf{R}$ such that $\Phi(f) = h \circ f \circ h^{-1}$ for all $f \in S$. We shall call h the *generating bijection* of the automorphism Φ.

3.3. Lemma. *Let h be a generating bijection of an automorphism Φ of S. Then h and h^{-1} are approximately continuous.*

Proof. We will prove approximate continuity of h at an arbitrary point $a \in \mathbf{R}$. Let $\varepsilon > 0$. Let $\beta: \mathbf{R} \to \mathbf{R}$ be an everywhere differentiable density-continuous function, existing by Lemma 2.1, such that

(i) $\beta(x) = 0$ if $|x| > \varepsilon$; $\beta(0) = 1$;

(ii) $\beta'(x) > 0$ if $-\varepsilon < x < 0$; $\beta'(x) < 0$ if $0 < x < \varepsilon$;

Define for $x \in \mathbf{R}$

$$g(x) = \beta(x - h(a)) + h(a). \tag{7}$$

Then $g \in S$. There is an $f \in S$ such that $\Phi(f) = g$. This implies that $h \circ f = g \circ f$. Since h is a bijection of $\mathbf{R}$ and $h(f(a)) = h \circ f(a) = g \circ h(a) = 1 + h(a)$, we must have $f(a) \neq a$. From the density-continuity of f, there exists a density-open set U such that $a \in U$ and $f(x) \neq a$ for every $x \in U$. The bijection h gives $g \circ h(x) = h \circ f(x) \neq h(a)$ for

$x \in U$. We infer from (5) and the above that, for $x \in U$, $\beta(h(x)-h(a)) \neq 0$, and therefore, $|h(x) - h(a)| < \varepsilon$. Thus h is approximately continuous at a.

Clearly, the bijcetion h is actually a homeomorphism of R in the natural topology. To establish approximate continuity of h^{-1} observe that $\Phi^{-1}(g) = h^{-1} \circ g \circ h$ for any $g \in S$.

3.4. Lemma. *Let $h: [0,1] \to [0,1]$ be an increasing homeomorphism (with respect to the natural topology) such that h^{-1} is not density-continuous at zero. Then there exists a differentiable density-continuous function $f: [0,1] \to [0,1]$ such that $h \circ f \circ h^{-1}$ is not density-continuous.*

Proof. As defined in [1], an interval set I at a point $x \in \mathbf{R}$ is a union of a sequence of disjoint nondegenerate closed intervals $[a_n, b_n]$, $n \in$ N, such that $x < \ldots < a_n < b_n < a_{n-1} < b_{n-1} < \ldots < a_1 < b_1$, and $\lim_{n\to\infty} a_n = x$.

Since h^{-1} is not density-continuous at zero, it does not preserve upper outer density of interval set at zero (see Theorem 2.13 in [15]). Let

$$I = \bigcup_{n\in\mathbf{N}} [a_n, b_n] \tag{8}$$

be an interval set such that $\overline{d}(I, 0) > 0$ and for

$$J = \bigcup_{n\in\mathbf{N}} [h^{-1}(a_n), h^{-1}(b_n)] \tag{9}$$

we have $d(J, 0) = 0$. Let $\{\alpha_n\}, \{\beta_n\}$ be sequences such that, for every $n \in$ N, $a_n < \alpha_n < \beta_n < b_n$ and

$$\overline{d}\Big(\bigcup_{n\in\mathbf{N}} [\alpha_n, \beta_n], 0\Big) > 0. \tag{10}$$

Then by (9)

$$d\Big(\bigcup_{n\in\mathbf{N}} [h^{-1}(\alpha_n), h^{-1}(\beta_n)], 0\Big) = 0. \tag{11}$$

For every $n \in$ N, let f_n be a density-continuous differentiable function given by Lemma 2.1 such that

(i) $f_n(x) = 0$ if $x \notin [h^{-1}(a_n), h^{-1}(b_n)]$; $f_n(x) = 2^{-n}\, h^{-1}(a_n)$ if $x \in [h^{-1}(\alpha_n), h^{-1}(\beta_n)]$;

(ii) $f_n'(x) > 0$ if $x \in \big(h^{-1}(a_n), h^{-1}(\alpha_n)\big)$;$f_n'(x) < 0$ if $x \in \big(h^{-1}(\beta_n), h^{-1}(b_n)\big)$;

Let $f\colon [0,1] \to [0,1]$ be defined as

$$f(x) = \sum_{n=1}^{+\infty} f_n(x). \tag{12}$$

Notice that $f(x) \neq 0$ implies that $x \in (h^{-1}(a_n), h^{-1}(b_n))$ for some $n \in \mathrm{N}$. Therefore $f(x) = 0$ except on a set having a dispersion point at zero and so f is density-continuous. Furthermore

$$0 \leq \frac{f(x)}{x} \leq 2^{-n} \tag{13}$$

for every $x \in (h^{-1}(a_n), h^{-1}(b_n))$, $n \in \mathrm{N}$. Thus f is differentiable at zero. It is obviously differentiable elsewhere.

Finally, even though $d\big(\bigcup_{n\in\mathrm{N}}[\alpha_n, \beta_n], 0\big) > 0$, the set

$$h \circ f \circ h^{-1}\Big(\bigcup_{n\in\mathrm{N}}[\alpha_n, \beta_n]\Big) = \bigcup_{n\in\mathrm{N}}\{h(2^{-n}h^{-1}(\alpha_n))\} \tag{14}$$

is countable and it does not contain zero; so $h \circ f \circ h^{-1}$ is not density-continuous at zero.

3.5. Corollary. *If h is a homeomorphism of* R *generating an automorphism Φ of S then h and h^{-1} are density-continuous.*

Proof. It follows immediately from 3.3 and 3.4 that h^{-1} must be density-continuous. Since, however, Φ^{-1} is generated by h^{-1}, h itself must be density-continuous.

3.6. Theorem. *All automorphisms of $\mathcal{C}_\mathcal{D}$ and $z\mathcal{C}_\mathcal{D}$ are inner.*

Proof. Notice that every homeomorphism of R is monotone, and thus differentiable almost everywhere, so the theorem easily follows from Lemma 3.3 and Corollary 3.5.

3.7. Remark. The class of homeomorphisms of $\mathcal{C}_\mathcal{D}$ and $z\mathcal{C}_\mathcal{D}$ is described in Theorem 2.13 of [15]. These are homeomorphisms h such that both h and h^{-1} preserve density points, or equivalently, both h and h^{-1} preserve upper outer density.

Also, it should be noted that we just proved that the semigroups $\mathcal{C}_\mathcal{D}$ and $z\mathcal{C}_\mathcal{D}$ have identical groups of automorphisms, even though the inclusion $z\mathcal{C}_\mathcal{D} \subset \mathcal{C}_\mathcal{D}$ is proper.

3.8. Lemma. *If Φ is an automorphism of $d\mathcal{C}_\mathcal{D}$ (or $a\mathcal{C}_\mathcal{D}$) and h is a homeomorphism of* R *generating it then h is differentiable.*

Proof. As we mentioned before, h is monotone, and thus differentiable almost everywhere. Let x_0 be a point of differentiability of h, and let $x \in \mathrm{R}$ be arbitrary.

Define $f(t) = t + x - x_0$ for $t \in \mathrm{R}$. Then f is differentiable and density continuous on R. Let δ be a positive number. We have

$$\frac{h(x+\delta)-h(x)}{\delta} = \frac{(h \circ f)(x_0+\delta) - (h \circ f)(x_0)}{\delta} = \frac{(\Phi(f) \circ h)(x_0+\delta) - (\Phi(f) \circ h)(x_0)}{\delta}. \tag{15}$$

But $\Phi(f)$ is an element of the semigroup of which Φ is an automorphism. If it is $d\mathcal{C}_{\mathcal{D}}$ that we are dealing with, then $\Phi(f)$ is differentiable, and h is differentiable at x_0, so that the difference quotient (15) has a limit, as $\delta \to 0$, equal to $\Phi(f)'(h(x_0))h'(x_0)$. For $a\mathcal{C}_{\mathcal{D}}$, since h must be density-continuous by Corollary 3.5, the composite $\Phi(f) \circ h$ is approximately differentiable at x_0 (see Theorem 1.3). Therefore h is approximately differentiable at x. Being a homeomorphism, it must be differentiable.

The following theorem is our analogue of the theorem proved by Kenneth Magill, Jr. in [11].

3.9. Theorem. *All automorphisms of $a\mathcal{C}_{\mathcal{D}}$ and $d\mathcal{C}_{\mathcal{D}}$ are inner.*

Proof. Corollary 3.5 tells us that automorphisms of $a\mathcal{C}_{\mathcal{D}}$ and $d\mathcal{C}_{\mathcal{D}}$ are given by density-continuous homeomorphisms. By Lemma 3.8 those homeomorphisms must be also differentiable.

It should be noted that the groups of automorphisms of $d\mathcal{C}_{\mathcal{D}}$ and $a\mathcal{C}_{\mathcal{D}}$ are identical, even though the inclusion $d\mathcal{C}_{\mathcal{D}} \subset a\mathcal{C}_{\mathcal{D}}$ is proper.

4. Possible alternate proof and acknowledgements.

Kenneth Magill, Jr., showed in [9] that, for a completely regular space X such that level sets $f^{-1}(\{x\})$ of points $x \in X$ for continuous selfmaps $f: X \to X$ form a subbase for the closed sets in X (X is then called *generated*) the semigroup of all continuous selfmaps has the inner automorphism property. We do not know, however, if the density topology is generated.

Query 1. Is the density topology generated?

As the density-closed G_δ sets form a base for the closed sets in the density topology, the answer to the above query would followe from the affirmative answer to the following:

Query 2. Given a G_δ subset E of $\mathbf{R}$ which is also density-closed, is there a density-continuous function $f:\mathbf{R} \to \mathbf{R}$ such that $E = f^{-1}(\{0\})$?

Notice that, by the result of Z. Zahorski in [21], one can find an approximately continuous f as in Query 2. Krzysztof Ciesielski, Lee Larson, and Krzysztof Ostaszewski in [5] were able to construct such an f, which is, in addition to the above, differentiable, for a closed set E.

When this work was started, the analytic structure of the density-continuous functions was not known well. The works of Andrew Bruckner ([1]) and Jerzy Niewiarowski ([13]) gave some insight to the density-continuous bijections of $\mathbf{R}$, while Krzysztof Ostaszewski ([14], [15], [16]) discussed density-continuity explicitly, but mostly in relation to approximate continuity and approximate differentiability. Since then, Krzysztof Ciesielski and Lee Larson in [4], the same two with the author in [5], and Maxim Burke in [3], have investigated the class of density-continuous functions more in depth.

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References:

1. Bruckner, A.M., Density-preserving homeomorphism and the theorem of Maximoff, *Quart. J. Math. Oxford*, (2)21 (1970), 337-347.
2. Bruckner, A.M., *Differentiation of Real Functions*, Lecture Notes in Mathematics, 659, Springer, 1978.
3. Burke, M., Some remarks on density-continuous functions, submitted.
4. Ciesielski, K., and Larson, L., The space of density-continuous functions, submitted.
5. Ciesielski, K., Larson L., and Ostaszewski, K., The structure of density-continuous functions, in preparation.
6. Gavrilov, M., On a semigroup of continuous functions, *Godishn. Sofiisk. Univ. Matem. Fac.*, (1964), 377-380.

7. Goffman, C., Neugebauer, C.J., and Nishiura, T., Density topology and approximate continuity, *Duke Math. J.*, (28)4 (1961), 497-505.
8. Goffman C., and Waterman, D., Approximately continuous transformations, *Proc. Amer. Math. Soc.*, 12 (1961), 116-121.
9. Magill, K.D. Jr., Another S-admissible class of spaces, *Proc. Amer. Math. Soc.*, 18 (1967), 295-298.
10. Magill, K.D., Jr., A survey of semigroups of continuous selfmaps, *Semigroups Forum*, 11 (1975-76), 189-282.
11. Magill, K.D., Jr., Automorphisms of the semigroup of all differentiable functions, *Glasgow Math. J.*, 8 (1967), 63-66.
12. Malcev, A.A., On a class of topological spaces, *Thesis for a report at the GSM*, Moscow, 1966, 23.
13. Niewiarowski, J., Density-preserving homeomorphisms, *Fund. Math.*, 106 (1980), 77-87.
14. Ostaszewski, K., Continuity in the density topology, *Real Anal. Exchange*, (7)2 (1982), 259-270.
15. Ostaszewski, K., Continuity in the density topology II, *Rend. Circ. Mat. Palermo*, (2)32 (1983), 398-414.
16. Ostaszewski, K., Density topology and the Luzin (N) condition, *Real Anal. Exchange*, 9 (1983-84), 390-393.
17. Oxtoby, J.C., *Measure and Category*, Springer, 1971.
18. Schreier, J., Über Abbildungen einer abstrakten Menge auf ihre Teilmengen, *Fund. Math.*, 28 (1937), 261-264.
19. Shneperman, L.B., Semigroups of continuous transformations, *Soviet Mathematics*, (3)3 (1962), 775-777.
20. Tall, F.D., The density topology, *Pacific J. Math.*, (62)1 (1976), 275-284.
21. Zahorski, Z., Sur la première dérivée, *Trans. Amer. Math. Soc.*, (69)1 (1950), 1-54.

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