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CONNECTIVITY FUNCTIONS $I^n \to I$ DENSE IN $I^n \times I$

Let X and Y be topological spaces and let $f: X \to Y$ be a function. Then f is said to be <u>connectivity function</u> provided that if C is a connected subset of X, then the graph of f restricted to C is a connected subset of $X \times Y$. Denote the graph of f restricted to C by $f|C$. The function f is defined to be <u>peripherally continuous</u> provided that for any $x \in X$ and any pair of open sets U and V containing x and $f(x)$, respectively, there exists an open set W such that $x \in W \subset U$ and $f(bd(W)) \subset V$ where $bd(W)$ is the boundary of W. The function f is <u>dense</u> in $X \times Y$ provided that if U and V are open subsets of X and Y, respectively, then there is a point $x \in U$ such that $f(x) \in V$. Clearly, the function is totally discontinuous (i.e., nowhere continuous) if Y is a non-degenerate Hausdorff space and f is dense in $X \times Y$.

For the definitions concerning simplexes and simplicial complexes the reader is referred to [2].

Examples of connectivity functions $I^2 \to I$ dense in I^3 where $I = [0,1]$ and hence totally discontinuous functions have been constructed in [1] and [3]. However, connectivity functions $I^n \to I$ dense in I^{n+1} have not been presented when $n > 2$. Toward this end we construct for $n \geq 2$ an example of a connectivity function $f: \Delta^n \to I$ dense in $\Delta^n \times I$ where Δ^n is an n-simplex by using a variation of a technique given in [3]. Now there exists a homeomorphism $g: I^n \to \Delta^n$, and hence it follows that $f \circ g: I^n \to I$ is a connectivity function dense in $I^n \times I$. Thus $f \circ g: I^n \to I$ is a totally discontinuous connectivity function.

Connectivity functions and peripherally continuous functions defined on certain spaces and in particular on I^n and Δ^n are equivalent whenever $n \geq 2$, [4].

Theorem 1. There exists a connectivity function $f: \Delta^n \to I$ dense in $\Delta^n \times I$ whenever $n \geq 2$.

Proof. Let $\Delta^n = v_0 v_1 \ldots v_n$ denote the unit n-simplex of Euclidean space R^{n+1} with vertices $v_0, v_1, \ldots, v_n$.

L_1: Let f be 0 on $\mathrm{bd}(\Delta^n)$ and 1 at the barycenter $p = v_0/(n+1) + v_1/(n+1) + \ldots + v_n/(n+1)$ of Δ^n. Let T be the set of all proper faces of Δ^n. Consider the triangulation of Δ^n given by the cone pT, and let L_1 be its $(n - 1)$-skeleton. The mesh of pT is $\sqrt{2}$. We can extend f linearly on each $(n - 1)$-simplex σ of pT because f is defined at all the vertices of σ. The variation of f on the boundary of each n-simplex in pT is ≤ 1.

$L_{m+1} (m \geq 1)$: Suppose that we have constructed a triangulation T' of Δ^n with L_m denoting its $(n - 1)$-skeleton, and suppose that we have defined f on the underlying polyhedron $|L_m|$ such that the following conditions hold:

(1) The mesh of T' is $\leq 2\sqrt{2}/(m+1)$.

(2) The function f is linear on each simplex of L_m.

(3) The variation of f on the boundary of each n-simplex in T' is $\le 1/m$.

(4) Each point of Δ^n is within $2\sqrt{2}/(m+1)$ of a point where f has value $(1 - (-1)^m)/2$.

We construct L_{m+1} and extend f to $|L_{m+1}|$ in the following way. If m is odd, let f be 0 at the barycenter p_i of each n-simplex σ_i of T'. But if m is even, let f be 1 at p_i. Now each point of Δ^n is within $2\sqrt{2}/(m+1)$ of a point where f has value $(1-(-1)^m)/2$. For each i, let T_i be the set of all proper faces of σ_i, and form the cone p_iT_i. Define a continuous function φ on each n-simplex $\sigma = u_0u_1...u_n$ of p_iT_i by
$\varphi(\alpha_0u_0 + \alpha_1u_1 +...+ \alpha_nu_n) = \alpha_0f(u_0) + \alpha_1f(u_1) + ...+ \alpha_nf(u_n)$ where $\alpha_k \ge 0$ and $\sum_{k=0}^{n} \alpha_k = 1$. Choose a positive integer N so that for each i, the variation of φ on the boundary of each n-simplex in the N^{th} barycentric subdivision K_i of p_iT_i is $\le 1/(m+1)$ and K_i has mesh $\le 2\sqrt{2}/(m+2)$. Then define f to be φ on the boundary of each n-simplex in $\cup K_i$. Let L_{m+1} be the (n-1)-skeleton of $\cup K_i$. The variation of f on the boundary of each n-simplex in $\cup K_i$ is $\le 1/(m+1)$.

$|L_m|$ is a subset of $|L_{m+1}|$ for each m. By construction, f is peripherally continuous on $\bigcup_{m=1}^{\infty} L_m$. Suppose $x \in \Delta^n - \bigcup_{m=1}^{\infty} |L_m|$. For every m, x lies in the interior of an n-simplex s_m such that as $m \to \infty$, $s_m \to x$ and the variation of f on $bd(s_m)$ approaches 0. If we choose $y_m \in bd(s_m)$, then $y_m \to x$. Let $f(x)$ be a cluster point of

$f(y_1), f(y_2), \ldots$. Then f is peripherally continuous at x. By construction, the graph of f is dense in $\Delta^n \times I$.

Theorem 2. There exist 2^c connectivity functions as defined in Theorem 1.

Proof. Let T_m be the set of $x \in \Delta^n - \bigcup_{k=1}^{\infty} |L_k|$ such that there exist n-simplexes Δ_0 and Δ_1 such that the diameters of Δ_0 and Δ_1 are less than $1/m$, $x \in \mathrm{int}(\Delta_0)$, $x \in \mathrm{int}(\Delta_1)$, $f(\mathrm{bd}(\Delta_0)) \subset [0,1/m)$, and $f(\mathrm{bd}(\Delta_1)) \subset (1 - 1/m, 1]$. Then on $G = \bigcap_{m=1}^{\infty} T_m$, we can define f to be either 0 or 1. By construction, G is dense in Δ^n.

If $x \in T_m$, then every point of $\mathrm{int}(\Delta_0) \cap \mathrm{int}(\Delta_1)$ is in T_m. So T_m is open. Therefore $\bigcap_{m=1}^{\infty} T_m$ is a G_δ-set. So G is a dense G_δ-subset of Δ^n on which the values of f can be chosen to be either 0 or 1.

By the Alexandrov theorem, every G_δ-subset of a complete space is homeomorphic to a complete space. Also every non-empty, complete, and dense-in-itself space contains the Cantor set topologically. Thus G has c elements. Thus the cardinality of the power set of G is 2^c. Now f can be arbitrarily defined on G to be 0 or 1. Hence for any $A \subset G$, we may let $f(A) = 0$ and $f(G - A) = 1$. Therefore there are 2^c connectivity functions $\Delta^n \rightarrow I$ as defined in Theorem 1.

Theorem 3. There exists a connectivity function $I^n \rightarrow I^k$ dense in $I^n \times I^k$ for any $n \geq 2$ and $k \geq 2$.

Proof. Let $f: I^n \rightarrow I$ be a connectivity function dense in $I^n \times I$, and let $h: I \rightarrow I^k$ be a Peano space-filling (continuous and onto) curve. Then $h \circ f: I^n \rightarrow I^k$ is a connectivity function [5] dense in $I^n \times I^k$.

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