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Extreme Path Derivatives of a Function with Respect to a Nonporous System of Paths

1. Introduction.

It is well known that the derivative of a differentiable function is in B_1, the first class of Baire, and that the extreme derivatives of an arbitrary function are in B_2, the second class of Baire. The Dini derivatives of a function in B_α are in $B_{\alpha+2}$ [7]. In recent years similar results have been obtained for approximate derivatives [8], [9], and approximate symmetric derivatives [6], etc..

Bruckner, O'Malley and Thomson [3] introduced the concept of path derivative as a unifying approach to the study of a number of generalized derivatives. Any generalized derivative for which the derivative at a point is a derive number of the function at that point falls into the framework of path derivatives. They noticed that much of the information concerning the behavior of a generalized derivative is contained in the geometry of the collection $E = \{E_x: x \in R\}$. One would like to have nice extreme path derivatives or path derivative for a nice function, i.e. a measurable (Borel measurable) $\overline{F}'_E$ for a measurable (Borel measurable) function F.

The behavior of path derivatives and extreme path derivatives depends on three factors: the thickness of the paths, the behavior of E_x and E_y when x and y are very close, and the behavior of the original function. In [2] we studied the Borel measurability of extreme path derivatives when the system of paths is continuous. In this paper we investigate the Borel measurability of extreme path derivatives when the paths are thick in some sense. Some familiar notions of measuring "thickness" of a set are outer density, density and porosity. Bruckner, O'Malley and Thomson in [3] showed that for a monotonic function F, and a nonporous system of paths E, the extreme path derivaties $\overline{F}'_E$ and $\underline{F}'_E$ are identical to the extreme derivatives $\overline{F}'$, and $\underline{F}'$ respectively. Thus Hajek's theorem [5] implies that $\overline{F}'_E$ and $\underline{F}'_E$ are functions in Baire class two. Similar results hold for a Lipschitz function with the Lipschitz constant M, since the function $F(x) - Mx$ is monotonic.

We begin with a preliminary section giving the basic definitions. In §3 we construct a Darboux Baire one function and a continuous function tailored to a nonporous system of paths with path derivative not being measurable and Borel measurable respectively. This shows that we can not drop the requirement that F is a Lipschitz function. We then continue by showing certain connections between the paths and the moduli of continuity of the function which still allows us to conclude that the extreme path derivatives are in Baire class two. Moduli of continuity are convenient tools to control the growth of functions. Here they are used to control the size of the gaps in paths.

2. Preliminaries.

2.1 Definition. Let $x \in [0,1]$. A path leading to x is a set $E_x \subset [0,1]$ such that $x \in E_x$ and x is a point of accumulation of E_x. A system of paths is a collection $E = \{E_x: x \in [0,1]\}$ such that each E_x is a path leading to x.

2.2 Definition. Let $F: [0,1] \longrightarrow R$ and let $E = \{E_x: x \in [0,1]\}$ be a system of paths. If $\lim\limits_{\substack{y\to x\\ y\in E_x}} \frac{F(y)-F(x)}{y-x} = f(x)$ is finite, then we say that F is E-differentiable at x and write $F'_E(x) = f(x)$. If F is E-differentiable at every point $x \in [0,1]$, then we say that F is E-differentiable.

The extreme E-derivatives of F are $\limsup\limits_{\substack{y\to x\\ y\in E_x}} \frac{F(y)-F(x)}{y-x}$ and $\liminf\limits_{\substack{y\to x\\ y\in E_x}} \frac{F(y)-F(x)}{y-x}$. When we are dealing with a specific system of paths tailored to a continuous function, we may alter $E = \{E_x: x \in [0,1]\}$ to $E_1 = \{\overline{E}_x: x \in [0,1]\}$, where $\overline{E}_x$ denotes the closure of E_x.

2.3 Definition. Let A be a subset of the real line, R, and let $x_0 \in R$. Define $P_+(A,x_0) = \limsup_{h\to 0^+} \frac{\lambda(A,x_0,h)}{h}$ where $\lambda(A,x_0,h)$ is the length of the largest open interval in $\tilde{A} \cap (x_0,x_0+h)$. ($\tilde{A}$ denotes the complement of A).

Similarly define $P_-(A,x_0)$ and let $P(A,x_0)$ be the larger of $P_+(A,x_0)$ and $P_-(A,x_0)$. Note that the numbers $P(A,x_0)$, $P_+(A,x_0)$, $P_-(A,x_0)$ all lie in the closed interval $[0,1]$ and that the larger $P(A,x_0)$ is, the larger are the gaps of A near x_0. The numbers $P_+(A,x_0)$, $P_-(A,x_0)$ and $P(A,x_0)$ are called porosity of A from the right, porosity of A from the left, and porosity of A at x_0 respectively.

2.4 Definition. Let $E = \{E_x: x \in [0,1]\}$ be a system of paths. (If E has any of these properties at each point, then we say that E has that property.)

(2.4.1) E is said to be unilateral at x if x is a unilateral point of accumulation of E_x.

(2.4.2) E is said to be nonporous from the left (right) at x if E_x has left (right) porosity 0 at x.

(2.4.3) E is said to be nonporous at x if E_x has porosity 0 at x.

(2.4.4) E is said to be bilateral at x if x is a bilateral point of accumulation of E_x.

2.5 Definition. A function ω will be called a modulus of continuity if ω is defined for positive reals, is increasing, and $\lim_{x\to 0^+}\omega(x) = 0$. A function $F \in C([0,1])$ will be said to have ω as its modulus of continuity and belongs to $C(\omega)$, if $x,y \in [0,1]$ implies $|F(x)-F(y)| < \omega(|x-y|)$. A modulus of continuity is called regular if $\omega(\lambda x) \le (\lambda+1)\omega(x)$ where $\lambda \in R^+$. (the positive reals). It readily follows from the definition of continuity that every $F \in C([0,1])$ is in $C(\omega')$ where ω' is defined as follows:

$$\omega'(\delta) = \sup_{|x-y|\le\delta} \{|F(x)-F(y)|\}.$$

It is clear that ω' is a regular modulus of continuity, and $\omega'(\delta) \le \omega(\delta)$, for all ω such that $F \in C(\omega)$. The modulus of continuity ω' is called the regular modulus of continuity corresponding to F. Note that the relations $F \in \mathrm{Lip}_M^\alpha$ and $\omega'(\delta) \le M\delta^\alpha$ are equivalent.

2.6 Definition. Let $F\colon R \longrightarrow R$ be a function, the oscillation of F at a is defined as follows:

$$\mathrm{OSC}(F;a) = \lim_{\delta\to 0}\left[\sup_{a-\delta\le z\le a+\delta} F(z) - \inf_{a-\delta\le z\le a+\delta} F(z)\right].$$

Throughout this paper $\tilde{A}$ denotes the complement of A and N is the set of positive integers.

3. Modulus of Continuity and Porosity.

Example 3.1 shows the existence of a Darboux Baire one function, and a nonporous system of paths E such that F'_E is not Lebesgue measurable. In example 3.2 we construct a continuous function and a nonporous system of paths so that F'_E is not Borel measurable. Remark 3.8 of [4] implies that F'_E may also be nonmeasurable.

Moduli of continuity are convenient tools to control the growth of functions. Here they are used to control the size of the gaps in paths. In Theorem 3.4 relations between moduli of continuity of a function and the sizes of the gaps are given. These relations guarantee that the extreme path derivatives are functions in Baire class two.

Example 3.1: There exist a Darboux Baire 1 function and a nonporous system of paths E such that F'_E exists everywhere, but it is not Lebesgue measurable.

Let P be a cantor like set with positive measure and $\tilde{P} = \bigcup_{n=1}^{\infty} (c_n, d_n)$. For each $n \in N$ choose sequences $\{a_{n,m}\}_{m=1}^{\infty}$, and $\{b_{n,m}\}_{m=1}^{\infty}$ with the following properties:

(i) $\lim_{m\to\infty} a_{n,m} = c_n$ $\qquad\qquad$ $\lim_{m\to\infty} b_{n,m} = d_n$,

(ii) $c_n < a_{n,m+1} < a_{n,m}$ $\qquad$ $b_{n,m} < b_{n,m+1} < d_n$,

(iii) $a_{n,m} - a_{n,m+1} < \frac{1}{n}(d_n - c_n)$ $\qquad$ $b_{n,m+1} - b_{n,m} < \frac{1}{n}(d_n - c_n)$,

and $\qquad a_{n,1} = b_{n,1} = \dfrac{d_n + c_n}{2}$.

On each subinterval $(a_{n,m+1}, a_{n,m})$ define the function

$$h_{n,m}(x) = \frac{1}{(x-a_{n,m+1})(a_{n,m}-x)} \sin\left[\frac{1}{(x-a_{n,m+1})(a_{n,m}-x)}\right]$$

$$h_{n,m}(a_{n,m}) = h_{n,m}(a_{n,m+1}) = 0.$$

Similarly define $h_{n,m}$ on each subinterval $[b_{n,m}, b_{n,m+1}]$. The function

$$F(x) = \begin{cases} 0 & \text{if } x \in P \\ h_{n,m}(x) & \text{if } x \in [a_{n,m+1}, a_{n,m}] \cup [b_{n,m}, b_{n,m+1}] \end{cases}$$

is Darboux Baire one. Let A be a nonmeasurable subset of P. Define the system of paths $E = \{E_x\colon x \in [0,1]\}$ as follows: $E_x = (x-\delta, x+\delta) \subset \tilde{P}$ for some positive δ for $x \in \tilde{P}$,

$$E_x = \{t\colon F(t) = t - x\} \quad \text{for } x \in P\backslash A,$$

$$E_x = \{t\colon F(t) = -t + x\} \quad \text{for } x \in A.$$

The system of paths E is nonporous and F'_E exists everywhere but F'_E is not measurable, since $F'_E = 1$ on $P\setminus A$, and $F'_E = -1$ on A.

Example 3.2: There exists a continuous function F and a nonporous system of paths $E = \{E_x: x \in [0,1]\}$ such that F'_E exists everywhere but it is not Borel measurable.

Let $\{p_n\}_{n=1}^{\infty}$ be an increasing sequence of positive real numbers tending to 1, so that

$$\lim_{n\to\infty} n(1 - p_n) = \infty.$$

We construct a Cantor like set as follows:

Remove an open interval $J_{1,1}$ of length p_1 from the middle part of the closed interval $[0,1]$. Call the remaining two closed intervals of equal length $I_{1,1}$ and $I_{1,2}$. Remove the open intervals $J_{2,1}$ and $J_{2,2}$ of length $p_2 \cdot |I_{1,1}|$ from the middle part of the closed intervals $I_{1,1}$ and $I_{1,2}$ respectively. Call the remaining four closed intervals of equal length $I_{2,1}$, $I_{2,2}$, $I_{2,3}$ and $I_{2,4}$. In general at the nth stage we remove open intervals $J_{n,1}, J_{n,2}, J_{n,3}, \ldots, J_{n,2^{n-1}}$ each of length $p_n \cdot |I_{n-1,1}|$ from the middle part of the 2^{n-1} closed intervals $I_{n-1,1}, I_{n-1,2}, \ldots, I_{n-1,2^{n-1}}$ remaining from the (n-1)th stage. Let P be the set which remains when this process has been carried out indefinitely.

The set P is perfect, nowhere dense, and has measure zero. Let I_1, I_2, I_3, ... be the sequence of intervals contiguous to P in $[0,1]$. Clearly $\lim_{n\to\infty} |I_n| = 0$. Let $I_n = (c_n, d_n)$. For each $n \in N$ choose sequences $\{a_{n,k}\}_{k=1}^{\infty}$, $\{b_{n,k}\}_{k=1}^{\infty}$ with the following properties:

$$a_{n,1} = b_{n,1} = \frac{d_n + c_n}{2}$$

$$c_n < a_{n,k+1} < a_{n,k} \qquad b_{n,k} < b_{n,k+1} < d_n$$

$$a_{n,k} - a_{n,k+1} < \frac{1}{n} |I_n| \qquad b_{n,k+1} - b_{n,k} < \frac{1}{n} |I_n|$$

$$\lim_{k\to\infty} a_{n,k} = c_n \qquad \lim_{k\to\infty} b_{n,k} = d_n$$

$$\lim_{k\to\infty} \frac{a_{n,k+1}}{a_{n,k}} = 1 \qquad \lim_{k\to\infty} \frac{b_{n,k+1}}{b_{n,k}} = 1.$$

We now define our function F on the interval $[0,1]$ so that each of the following is true:

(i) F is continuous on $[0,1]$,

(ii) F vanishes on $P \bigcup \{\{a_{n,k}\}_{k=1}^{\infty} \bigcup \{b_{n,k}\}_{k=1}^{\infty}\}_{n=1}^{\infty}$,

(iii) F is differentiable on each I_i,

(iv) $F\left(a_{n,k+1} + \frac{a_{n,k} - a_{n,k+1}}{4}\right) = -\sqrt{a_{n,k} - c_n}$,

$$F\left(a_{n,k+1} + \frac{3(a_{n,k} - a_{n,k+1})}{4}\right) = \sqrt{a_{n,k} - c_n}\ ,$$

$$F\left(b_{n,k} + \frac{b_{n,k+1} - b_{n,k}}{4}\right) = -\sqrt{d_n - b_{n,k}}\ ,$$

$$F\left(b_{n,k} + \frac{3(b_{n,k+1} - b_{n,k})}{4}\right) = \sqrt{d_n - b_{n,k}}\ , \quad n = 1,2,3,\dots\ .$$

Choose a non-Borel measurable subset A of P. We construct a system of paths $E = \{E_x: x \in [0,1]\}$ in the following manner. For any $x \in A$, define:

$$E_x = \{t: F(t) = t - x\}.$$

For $x \in P\setminus A$, define

$$E_x = \{t: F(t) = -t + x\},$$

and for x not in P, choose a positive number δ small enough so that $[x-\delta, x+\delta] \cap P = \phi$. In this case we define $E_x = [x-\delta, x+\delta]$. F is a continuous function defined on $[0,1]$, and the derivative function F'_E exists everywhere, and $F'_E = 1$ on A, and -1 on $P\setminus A$.

We verify the fact that each path E_x is nonporous on both sides at x. Let us compute the porosity $P^+(E_x, x)$ for each x; similar arguments may be applied to obtain the left porosity.

For points x in $[0,1]\backslash P$ it is trivially true that $P^+(E_x,x) = 0$. Let x be a point of A. If x is a bilateral point of accumulation of P, let us estimate the size of $\lambda(E_x,x,h)$ where h is a positive real number. If (a,b) is the largest subinterval of $(x,x+h)$ that is disjoint from the set E_x, then because F is continuous either

$$F(t) > t - x \quad \text{everywhere on} \quad (a,b)$$

or

$$F(t) < t - x \quad \text{everywhere on} \quad (a,b).$$

Consider the former situation. For this to be the case (a,b) must be a subinterval of some interval $(a_{i,k+1},a_{i,k}) \subset I_i$ or $(b_{i,k},b_{i,k+1}) \subset I_i$. Then

$$\frac{\lambda(E_x,x,h)}{h} \le \frac{(a_{i,k} - a_{i,k+1})}{h}$$

$$\le \frac{a_{i,k} - a_{i,k+1}}{(a_{i,k+1} - x)} \le \frac{\frac{1}{i}\,|I_i|}{(1 - p_i)|I_i|} \le \frac{1}{i(1 - p_i)}$$

and since i tends to infinity as h tends to zero, this is arbitrarily small for h sufficiently close to zero. Alternatively, let us consider the latter situation. E_x misses the subinterval $(c_n,a_{n,k})$ for which $\sqrt{a_{n,k} - c_n} < (a_{n,k} - x)$. So $(a_{n,k} - c_n) < (a_{n,k} - x)^2$. Also E_x misses the subinterval $(b_{n',k'},d_n)$ of $I_{n'}$ for which $\sqrt{d_{n'} - b_{n',k'}} < (d_{n'} - x)$.

Thus $(d_{n'} - b_{n,k'}) < (d_{n'} - x)^2$.

In this case $(b - a) = \lambda(E_x,x,h) \le (a_{n,k} - x)^2 + (d_{n'} - x)^2 + (1 - p_n)h$ when $c_n \le x + h \le d_n$ and $(d_{n'} - x) \le h$. We have

$$\frac{\lambda(E_x,x,h)}{h} \le \frac{(a_{n,k} - x)^2 + (d_{n'} - x)^2 + (1 - p_n)h}{h}$$

$$\le (d_{n'} - x) + (a_{n,k} - x) + (1 - p_n).$$

For small h, this expression is small too, and we have verified that $P^+(E_x,x) = 0$. Similarly $P^+(E_x,x) = 0$ when $x = d_n$ for some n. When $x = c_n$, E_x has at least one point in every subinterval $(a_{n,k+1},a_{n,k})$, so $\lambda(E_x,x,h) \le (a_{n,k} - a_{n,k+1})$

$$\frac{\lambda(E_x,x,h)}{h} \le \frac{a_{n,k} - a_{n,k+1}}{a_{n,k+1}}$$

which could be as small as we please, as h tends to zero. So $P^+(E_x,x) = 0$ for every $x \in A$.

Similarly $P^+(E_x,x) = 0$ for every $x \in P\backslash A$. The derivative function F'_E is not Borel measurable since $\{x: F'_E(x) \ge 1/2\} \cap P = A$ which is not a Borel measurable set.

Lemma 3.3: Let F be a function defined on $[0,1]$, ϵ a positive real number, and $OSC(F;a) < \epsilon$ for each $a \in [0,1]$. Then there exists a continuous function g so that $|F(x) - g(x)| < 2\epsilon$.

Proof: For each $a \in [0,1]$, since $OSC(F,a) < \epsilon$, there exists a $\delta_a > 0$ such that for all $t, z \in (a-\delta_a, a+\delta_a)$, $|F(t) - F(z)| < \epsilon$. As the open intervals $(a-\delta_a, a+\delta_a)$ form an open covering of $[0,1]$, there exists a finite number of points $a_i \in [0,1]$ such that the open intervals $(a_i-\delta a_i, a_i+\delta_i)$ form a covering of $[0,1]$. Without loss of generality we suppose $0 = a_1 < a_2 < a_3 < \ldots < a_n = 1$. Define

$$H_i(x) = \frac{F(a_{i+1}) - F(a_i)}{a_{i+1} - a_i}(x-a_i) + F(a_i)$$

for $i = 1,2,\ldots,n-1$.

Let $g(x) = H_i(x)$ where $x \in [a_i, a_{i+1}]$ for $i = 1,2,\ldots,n-1$.

The function g is a continuous function and for each $x \in [0,1]$, $|g(x) - F(x)| \leq 2\epsilon$.

Theorem 3.4: Let $E = \{E_x: x \in [0,1]\}$ be a bilateral system of paths and $\{a_n\}_{n=1}^{\infty}$ be a decreasing positive sequence tending to zero so that $E_x \cap [x-a_n, x-a_{n+1}]$ and $E_x \cap [x+a_{n+1}, x+a_n]$ are nonempty for each $x \in [0,1]$. Also let $F \in C(\omega) \subset C([0,1])$. Suppose the following:

(i) $\omega(a_n) \bullet [1/a_{n+1} - 1/a_n]$ tends to zero.

(ii) $\omega(a_n - a_{n+1})/a_{n+1}$ tends to zero.

Then

a) if the function F is E-differentiable, $F'_E \in B_1$,

b) $\overline{F}'_E \in B_2$, $\underline{F}'_E \in B_2$.

Proof: It is enough to prove the theorem for the right upper E-derivative. Let $F_n(x) = \sup\{(F(t)-F(x))/(t-x)\colon\ t \in E_x \cap [x+a_{n+1}, x+a_n]\}$. By continuity of $F(x)$, for every positive ϵ, a positive δ could be found so that $|F(x) - F(y)| < \epsilon$ when $|x-y| < \delta$. Without loss of generality assume that each E_x is a closed set, thus $F_n(x) = (F(s)-F(x))/(s-x)$ for some $s \in E_x \cap [x+a_{n+1}, x+a_n]$. For $|y-x| < \delta$, $F_n(y) = (F(t)-F(y))/(t-y)$ for some $t \in E_y \cap [y+a_{n+1}, y+a_n]$, so

$$|F_n(x) - F_n(y)| = \left|\frac{F(s) - F(x)}{s-x} - \frac{F(t) - F(y)}{t-y}\right| =$$

$$\left|\frac{F(s) - F(x)}{s-x} - \frac{F(t) - F(s) + F(s) - F(x) + F(x) - F(y)}{t-y}\right|$$

$$\le |F(s) - F(x)| \bullet \left|\frac{1}{s-x} - \frac{1}{t-y}\right| + \left|\frac{F(t) - F(s)}{t-y}\right| + \left|\frac{F(x) - F(y)}{t-y}\right|.$$

But $a_{n+1} \le s - x \le a_n$, and $a_{n+1} \le t - y \le a_n$. Hence

$$\left|\frac{1}{s-x} - \frac{1}{t-y}\right| < \frac{1}{a_{n+1}} - \frac{1}{a_n} .$$

So

$$|F_n(x) - F_n(y)| \le \omega(a_n)\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right) + \frac{\omega(a_n - a_{n+1} + 2\delta)}{a_{n+1}} + \frac{\epsilon}{a_{n+1}}$$

when $|x-y| \le \delta$, which implies

$$OSC(F_n;x) = \lim_{\delta\to 0} [\max_{x-\delta\le z\le x+\delta} F(z) - \min_{x-\delta\le z\le x+\delta} F(z)]$$

$$\le \omega(a_n)\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right) + \frac{\omega(a_n - a_{n+1})}{a_{n+1}} = \frac{1}{2}\gamma_n .$$

The assumptions (i) and (ii) imply that γ_n tends to zero, and by lemma 3.3, for each $n \in N$, a continuous function g_n could be found so that $|F_n(x) - g_n(x)| < \gamma_n$ for every $x \in [0,1]$.

When the function F is E-differentiable for each $x \in [0,1]$ we have

$$F'_E(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} g_n(x); \quad \text{so} \quad F'_E \in B_1 .$$

In general $\overline{F}'_E(x) = \limsup_{n\to\infty} F_n(x) = \limsup_{n\to\infty} g_n(x)$; so $\overline{F}'_E \in B_2$. Similarly $\underline{F}'_E$ is a function in Baire class two.

<u>Remark 3.5</u>: For ω, a regular modulus of continuity, we have

$0 \le \omega((a_n - a_{n+1})/a_{n+1}) \le (1/a_{n+1} + 1)\omega(a_n - a_{n+1}) = \omega(a_n - a_{n+1})/a_{n+1} + \omega(a_n - a_{n+1})$.

Since $\{a_n\}$ is a positive sequence tending to zero, if $\omega(a_n - a_{n+1})/a_{n+1}$ tends to zero, $\omega((a_n - a_{n+1})/a_{n+1})$ vanishes as n gets large, which implies $\lim_{n\to\infty} a_n/a_{n+1} = 1$.

For a system of paths $E = \{E_x : x \in [0,1]\}$ such that $E_x \cap [x+a_{n+1}, x+a_n] \neq \phi$ for all natural numbers n, we have

$$0 \le P_+(E_x,x) = \limsup_{h\to 0} \frac{\lambda(E_x,x,h)}{h} \le \lim_{n\to\infty} \left(-1 + \frac{a_n}{a_{n+1}}\right) = 0,$$

which implies E_x is nonporous from the right at x.

<u>Remark 3.6</u>: When $\omega(\delta) \ge \sqrt{\delta}$, and $\omega(a_n - a_{n+1})/a_{n+1}$ tends to zero, the quantity $(1/a_{n+1} - 1/a_n)$ tends to zero as n tends to infinity.

<u>Proof</u>: For each natural number n, $(a_n - a_{n+1})^2 \ge 0$. Therefore

$$\left(\frac{a_n}{a_{n+1}}\right)^2 \ge \frac{2a_n - a_{n+1}}{a_{n+1}}$$

which implies

$$\frac{a_n}{a_{n+1}^2} - \frac{1}{a_{n+1}} \geq \frac{2a_n - a_{n+1}}{a_n a_{n+1}} - \frac{1}{a_{n+1}} = \frac{a_n - a_{n+1}}{a_n a_{n+1}} = \frac{1}{a_{n+1}} - \frac{1}{a_n} .$$

Since the function $F(x) = \sqrt{x}$ is increasing for $x > 0$,

$$\sqrt{\frac{a_n}{a_{n+1}^2} - \frac{1}{a_{n+1}}} \geq \sqrt{\frac{1}{a_{n+1}} - \frac{1}{a_n}} .$$

Thus

$$\frac{\omega(a_n - a_{n+1})}{a_{n+1}} \geq \frac{\sqrt{a_n - a_{n+1}}}{a_{n+1}} \geq \sqrt{\frac{1}{a_{n+1}} - \frac{1}{a_n}} \geq 0.$$

If $\dfrac{\omega(a_n - a_{n+1})}{a_{n+1}}$ tends to zero, then $\lim\limits_{n\to\infty} \left(\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n}\right) = 0.$

<u>Remark 3.7</u>: If $\omega(\delta) \geq \delta^{\alpha}$ for some α, $0 < \alpha < 1$, and $\{a_n\}_{n=1}^{\infty}$ is a decreasing positive sequence tending to zero, with $[\omega(a_n)]^{\alpha} \leq a_n^{\alpha}/(a_{n+1})^{1-\alpha}$, then

$$\omega(a_n)\left[\frac{1}{a_{n+1}} - \frac{1}{a_n}\right] \text{ tends to zero, if } \frac{\omega(a_n - a_{n+1})}{a_{n+1}} \text{ tends to zero.}$$

<u>Proof</u>: Since $\omega(\delta) \geq \delta^{\alpha}$ for some α,

$$\frac{\omega(a_n - a_{n+1})}{a_{n+1}} \geq \frac{(a_n - a_{n+1})^\alpha}{a_{n+1}} = \frac{a_n^\alpha a_{n+1}^\alpha}{a_{n+1}} \left[\frac{1}{a_{n+1}} - \frac{1}{a_n}\right]^\alpha$$

$$= \frac{a_n^\alpha}{(a_{n+1})^{1-\alpha}} \left[\frac{1}{a_{n+1}} - \frac{1}{a_n}\right]^\alpha \geq (\omega(a_n))^\alpha \left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right)^\alpha$$

$$\geq \left[\omega(a_n)\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right)\right]^\alpha,$$

which implies $\omega(a_n)\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right)$ tends to zero as $\frac{\omega(a_n - a_{n+1})}{a_{n+1}}$ tends to zero.

Lemma 3.8: If ω is a regular modulus of continuity, then the function $\omega^*(t) = t \inf\limits_{0<x\leq t} \frac{\omega(x)}{x}$ is a modulus of continuity such that $\frac{\omega^*(t)}{t}$ is nonincreasing, and $\frac{1}{2}\,\omega(t) \leq \omega^*(t) \leq \omega(t)$.

Proof: It is clear that $\omega^*(t)$ is defined for $t > 0$ and $\frac{\omega^*(t)}{t}$ is nonincreasing. If $t_1 < t_2$, and

$$\inf_{0<x\leq t_2} \frac{\omega(x)}{x} < \inf_{0<x\leq t_1} \frac{\omega(x)}{x},$$

then

$$\inf_{0<x\leq t_2} \frac{\omega(x)}{x} = \inf_{t_1\leq x\leq t_2} \frac{\omega(x)}{x}$$

and hence $\omega^*(t_2) = \inf_{t_1 \le x \le t_2} (t_2/x)\omega(x) > \omega(t_1) \ge \omega^*(t_1)$. If $t_1 < t_2$, and

$\inf_{0<x\le t_2} \omega(x)/x = \inf_{0<x\le t_1} \omega(x)/x$, then

$$\omega^*(t_2) = t_2 \inf_{0<x\le t_2} \frac{\omega(x)}{x} > t_1 \inf_{0<x\le t_1} \frac{\omega(x)}{x} = \omega^*(t_1).$$

Thus $\omega^*(t)$ is increasing.

Since $\omega^*(t) \le \omega(t)$

$$\lim_{t\to 0} \omega^*(t) = 0.$$

For $0 < x \le t$, let $\lambda = t/x$. Then

$$\omega(t) = \omega\left(\frac{t}{x} \bullet x\right) = \omega(\lambda x) \le (\lambda+1)\omega(x) \le 2\lambda\omega(x).$$

Hence

$$\frac{\omega(t)}{t} \le 2\frac{\omega(x)}{x}.$$

Thus

$$\omega^*(t) = t \inf_{0<x\le t} \frac{\omega(x)}{x} \ge t \bullet \frac{1}{2}\left(\frac{\omega(t)}{t}\right) = \frac{1}{2}\omega(t).$$

Therefore

$$\frac{1}{2}\omega(t) \le \omega^*(t) \le \omega(t),$$

and consequently the moduli of continuity ω and ω^* have the same order of convergence to zero as t tends to infinity.

Remark 3.9: For a regular modulus of continuity ω, since

$$0 \le \frac{1}{2}\omega(a_n)\left(\frac{1}{a_{n+1}} - \frac{1}{a_n}\right) = \frac{\frac{1}{2}\omega(a_n)}{a_n}\left(\frac{a_n}{a_{n+1}} - 1\right)$$

$$\le \frac{\omega^*(a_n)}{a_n}\left(\frac{a_n}{a_{n+1}} - 1\right) \le \frac{\omega^*(a_n-a_{n+1})}{a_n-a_{n+1}}\left(\frac{a_n-a_{n+1}}{a_{n+1}}\right)$$

$$= \frac{\omega^*(a_n-a_{n+1})}{a_{n+1}} \le \frac{\omega(a_n-a_{n+1})}{a_{n+1}} ,$$

we omit the condition (i) of Theorem 3.4.

Corollary 3.10: Let F be a function of class Lipschitz α for some $0 < \alpha < 1$ and let E be a system of paths. Suppose for all $n \in N$, $E_x \cap [x - 1/n^\alpha, x - 1/(n+1)^\alpha] \neq \phi$ and $E_x \cap [x + 1/(n+1)^\alpha, x + 1/n^\alpha] \neq \phi$ for each $x \in [0,1]$. Then

1) if F is E-differentiable, then $F'_E \in B_1$.

2) $\overline{F}'_E \in B_2$, $\underline{F}'_E \in B_2$.

Proof: In order to fulfill the conditions of Theorem 3.4, we should show that $\omega\left[\frac{1}{n^{\alpha}}\right]((n+1)^{\alpha}-n^{\alpha})$ and $(n+1)^{\alpha}\omega\left[\frac{-1}{(n+1)^{\alpha}}+\frac{1}{n^{\alpha}}\right]$ tends to zero as n tends to infinity. Since $\lim_{n\to\infty}((n+1)^{\alpha} - n^{\alpha}) = 0$ when $0 < \alpha < 1$, $\omega\left(\frac{1}{n^{\alpha}}\right)((n+1)^{\alpha} - n^{\alpha})$ tends to zero.

Thus
$$(n+1)^{\alpha}\omega\left(\frac{1}{n^{\alpha}} - \frac{1}{(n+1)^{\alpha}}\right) \le M(n+1)^{\alpha}\left(\frac{1}{n^{\alpha}} - \frac{1}{(n+1)^{\alpha}}\right)^{\alpha}$$
$$= M\left(\frac{n+1}{n}\right)^{\alpha}\left(\frac{n}{n^{\alpha}} - \frac{n}{(n+1)^{\alpha}}\right)^{\alpha}.$$

Since
$$\lim_{n\to\infty}\left[\frac{n}{n^{\alpha}} - \frac{n}{(n+1)^{\alpha}}\right] = \lim_{n\to\infty}\frac{n}{n^{\alpha}}\left(1 - \frac{1}{\left(\frac{n+1}{n}\right)^{\alpha}}\right)$$
$$= \lim_{n\to\infty}\frac{1 - \left(\frac{n}{n+1}\right)^{\alpha}}{n^{\alpha-1}} = \lim_{n\to\infty}\frac{-\alpha\left(\frac{n}{n+1}\right)^{\alpha-1}\left(\frac{-1}{(n+1)^{2}}\right)}{(\alpha-1)n^{\alpha-2}}$$
$$= \lim_{n\to\infty}\frac{\alpha}{\alpha-1}\left(\frac{n(n+1)^{1-\alpha}}{(n+1)^{2}}\right) = \frac{\alpha}{\alpha-1}\lim_{n\to\infty}\frac{n(n+1)^{1-\alpha}}{(n+1)^{2}} = 0,$$
$$\lim_{n\to\infty}(n+1)^{\alpha}\omega\left(\frac{1}{n^{\alpha}} - \frac{1}{(n+1)^{\alpha}}\right) = 0.$$

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