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DIMENSION PRINTS IN EUCLIDEAN SPACE

We summarize the results given in a recent paper [4] and give some diagrams that were not given there. With each set S in $\mathbb{R}^n$ we associate a second set $P(S)$, again in $\mathbb{R}^n$, called the dimension print of S. These dimension prints are designed to give additional information concerning the fractal nature of a fractal set S that cannot be obtained solely from a knowledge of the Hausdorff dimension of S.

In order to define the dimension print of a set, we first have to introduce a family of measures generalizing the Hausdorff measures (see, for example, [3]). We introduce a covering class $\mathcal{B}$ of "boxes" B that are rectangular parallelepipeds that do not, in general, have their edges parallel to the cordinate axes. When B is a typical box in $\mathcal{B}$ we use

$$\ell_1 \geq \ell_2 \geq \dots \geq \ell_n > 0,$$

to denote the edge-lengths of B, taken in decreasing order. For each non-negative vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, we introduce the measure μ^α defined by

$$\mu^\alpha(S) = \sup_{\delta>0} \inf \{ \sum_{i=1}^{\infty} (\ell(B_i))^\alpha : B_i \in \mathcal{B}, \text{diam } B_i \leq \delta, \bigcup_{i=1}^{\infty} B_i \supset S \},$$

using the multi-index notation wherein

$$\ell^\alpha = \ell_1^{\alpha_1} \ell_2^{\alpha_2} \dots \ell_n^{\alpha_n}.$$

The dimension print of S is then the set $P(S)$ of all non-negative vectors α with

$$\mu^{\alpha}(S) > 0.$$

It follows immediately, from this definition, that

$$P\left(\bigcup_{i=1}^{\infty} S_i\right) = \bigcup_{i=1}^{\infty} P(S_i),$$

for each sequence $S_1, S_2, \ldots$ of sets in $\mathbb{R}^n$.

We give illustrations of the dimension prints of a square, a cube, a cartesian product of certain regular Cantor sets, a circle, a sphere, a circular cylinder, and a twisted cubic curve. These sets are regarded, for this purpose, as sets lying in $\mathbb{R}^3$. The dimension print of a linear set L, regarded as a set in $\mathbb{R}^3$, is just an interval lying along the α_1-axis with length equal to the Hausdorff dimension of L. The dimension print of a planar set P, regarded as a set in $\mathbb{R}^3$, is just a copy, lying in the (α_1,α_2)-plane of $\mathbb{R}^3$, of the two dimensional dimension print of P, regarded as a set in $\mathbb{R}^2$. All the diagrams are on the same scale as orthogonal projections onto the plane $\alpha_1 + \alpha_2 + \alpha_3 = 0$. In each case we give a set of inequalities defining the dimension print as a subset of the positive octant, and also the successive dimensions of the set. These successive dimensions are just the intercepts that the dimension print cuts from the coordinate axes; the first is the Hausdorff dimension of the set, the sequence is always decreasing, but the decrease at each stage cannot exceed 1.

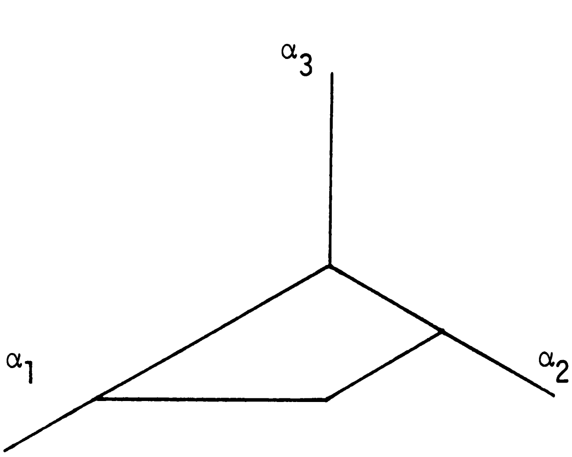

Figure 1. The dimension print of a square is defined by $\alpha_1 + \alpha_2 \leq 2$, $\alpha_2 \leq 1$, $\alpha_3 = 0$; the successive dimensions are 2,1,0.

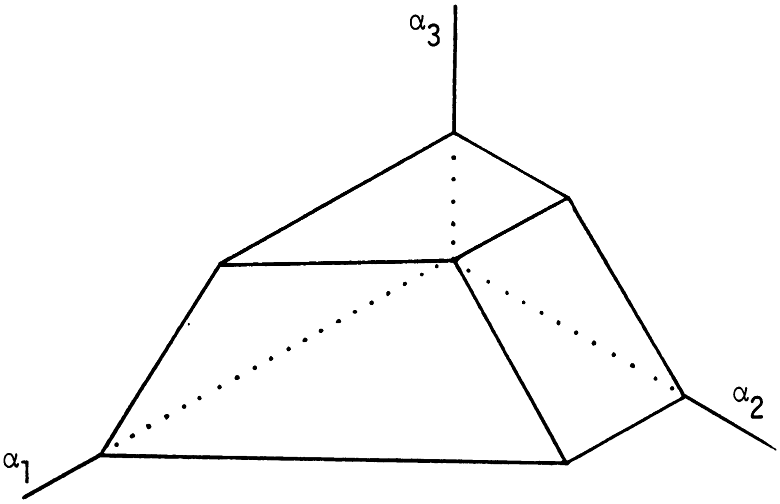

Figure 2. The dimension print of a cube is defined by $\alpha_1 + \alpha_2 + \alpha_3 \leq 3$, $\alpha_2 + \alpha_3 \leq 2$, $\alpha_3 \leq 1$; the successive dimensions are 3,2,1.

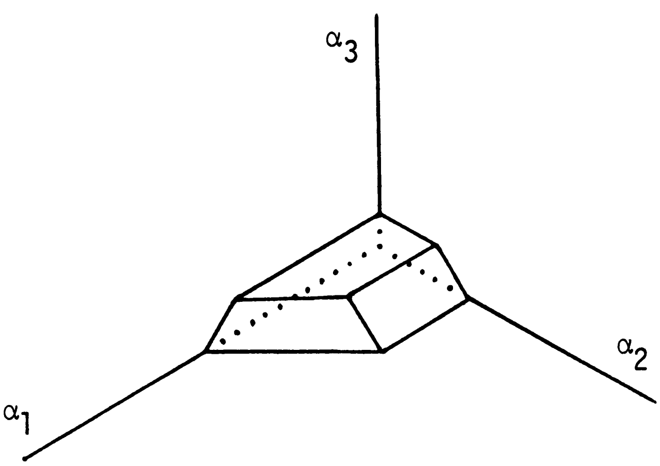

Figure 3. The dimension print of the cartesian product of three linear regular Cantor sets of dimensions $d_1 \geq d_2 \geq d_3$ is defined by $\alpha_1 + \alpha_2 + \alpha_3 \leq d_1 + d_2 + d_3$, $\alpha_2 + \alpha_3 \leq d_2 + d_3$, $\alpha_3 \leq d_3$; the successive dimensions are $d_1 + d_2 + d_3$, $d_2 + d_3$, d_3. The figure is drawn for the case when $d_1 = \frac{3}{4}$, $d_2 = \frac{1}{2}$, $d_3 = \frac{1}{4}$.

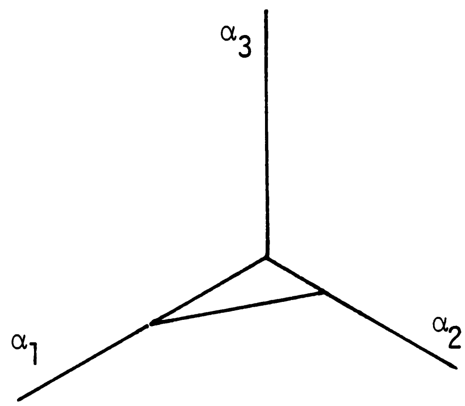

Figure 4. The dimension print of a circle is defined by $\alpha_1 + 2\alpha_2 \leq 1$, $\alpha_3 = 0$; the successive dimensions are $1, \frac{1}{2}, 0$.

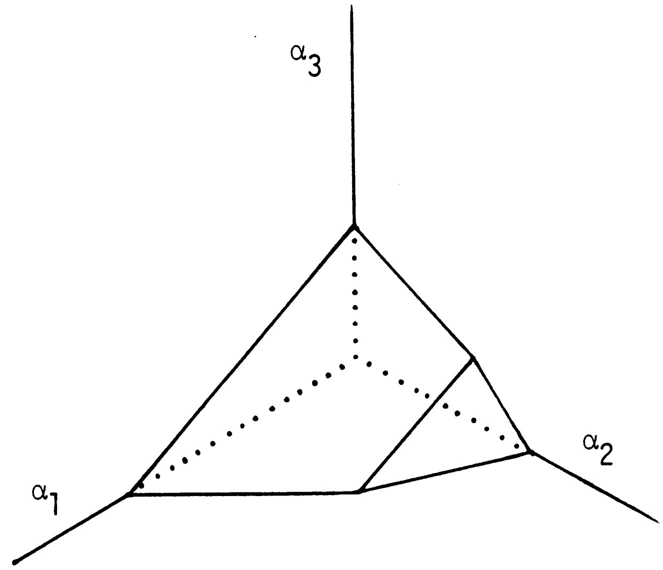

Figure 5. The dimension print of a sphere is defined by $\alpha_1 + \alpha_2 + 2\alpha_3 \leq 2$, $\alpha_1 + 2\alpha_2 + 2\alpha_3 \leq 3$; the successive dimensions are $2, 1\frac{1}{2}, 1$.

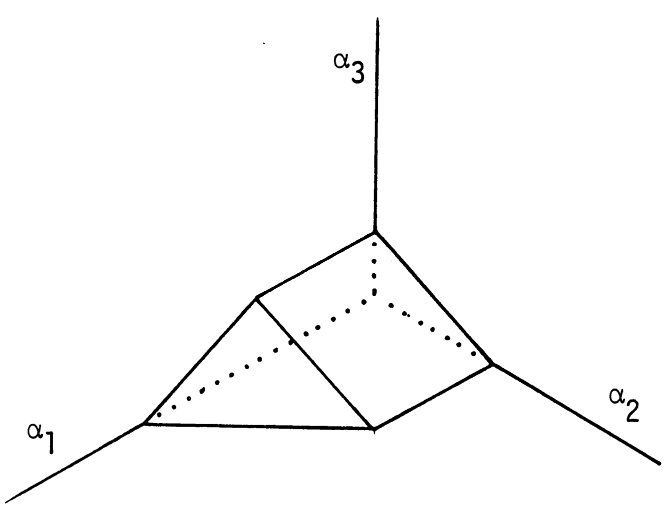

Figure 6. The dimension print of a cylinder is defined by $\alpha_1 + \alpha_2 + 2\alpha_3 \leq 1$, $\alpha_2 + 2\alpha_3 \leq 1$; the successive dimensions are $2, 1, \frac{1}{2}$.

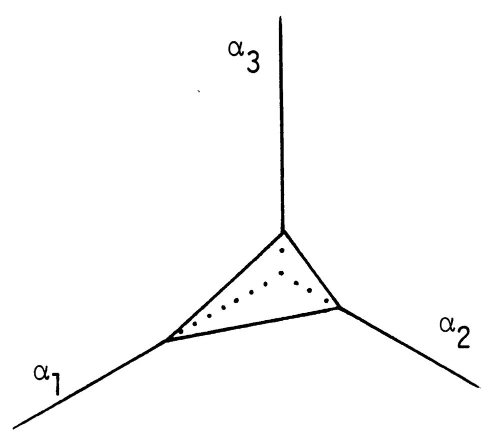

Figure 7. The dimension print of a twisted cubic curve is defined by $\alpha_1 + 2\alpha_2 + 3\alpha_3 \leq 1$, the successive dimensions are $1, \frac{1}{2}, \frac{1}{3}$.

Although our examples are of very simple sets with very simple dimension prints, much more complicated examples with more complicated dimension prints can be built up by taking countable unions of copies of known examples. The dimension print will, in general, be neither convex nor polyhedral, but it will always have the origin as a star centre.

If a set S can be covered by a countable family of congruent copies of one of its subsets, say T, then T has the same dimension print as S. Thus, for example, any non-empty relatively open subset of a sphere will have the same dimension print as the sphere.

The examples have certain stability properties under suitably small C^{∞} deformations. If the cube, the sphere or the twisted cubic curve are subjected to such deformations, their dimension prints will not change. If the square or circle are subjected to such deformations that leave them as planar sets, then again, the dimension prints will not change. If the cylinder is subjected to such a deformation that leaves a developable surface, the dimension print will not change.

One should be able to confirm or refute some of one's intuitive impressions of fractal sets by calculating their dimension prints. For example, the Lorenz attractor (see, for example, [2]) appears locally to be like the union of an uncountable family of twisted cubic curves, the local cross-section being a Cantor dust, in the sense of Mandelbrot [1], with a small Hausdorff dimension. So the dimension print of the Lorenz attractor must contain that of the twisted cubic curve, but should not be a much larger set. Similar considerations might well apply to any fractal set chosen to model cirrus clouds. One approach to the theory of turbulence is based on the idea that after a vortex sheet has developed, it becomes unstable and rolls up only to suffer similar instabilities on smaller and smaller scales. A model might well assume that the vorticity becomes concentrated on a fractal set with a dimension print including and rather similar to that of the sphere.

REFERENCES

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