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MULTIPLE GENERALISED RIEMANN INTEGRALS ON COMPACT INTERVALS AND APPLICATIONS

1. A quest for nonuniformity

If I is a compact interval in R^n with the norm $|x| = \max_i |x_i|$, and if $f : I \to R^p$ is a given mapping, a Riemann-type integration for f on I requires the use of P-partitions $\Pi = \{(x^1, I^1), \ldots, (x^q, I^q)\}$ of I, corresponding Riemann sums $S(f,\Pi) = \sum_{j=1}^{q} m(I^j) f(x^j)$ (with $m(I^j)$ the usual volume of I^j), and a way to control how the I^j shrink to make the Riemann sums converge to the value of the integral. Given a positive function (or gauge) δ on I, Π is said to be δ-fine if $I^j \subset B[x^j, \delta(x^j)]$ $(1 \leq j \leq q)$, and J is called the Kurzweil-Henstock integral of f over I if for each $\varepsilon > 0$, one can find a gauge δ on I such that

$$|S(f,\Pi) - J| \leq \varepsilon \tag{1}$$

whenever Π is δ-fine ([2],[4]). The usual Riemann integral corresponds to the special case of requiring constant gauges in the above definition and the benefit of abandoning, in the Riemann definition, following Kurzweil and Henstock, this uniformity requirement in the size of the shrinking intervals, is the obtention of an integral more general than Lebesgue's one and equivalent, when n = 1, to the Denjoy-Perron integral. Thus, the Kurzweil-Henstock integral integrates, when n = 1, every derivative.

The multiple Kurzweil-Henstock integral is not powerful enough to integrate the divergence of any differentiable vector field on I, with a formula corresponding to Stokes theorem, but this may be achieved if we abandon, in the Kurzweil-Henstock definition, the uniformity requirement in the shape of the shrinking intervals. The various ways of doing it depend upon the introduction of a concept of irregularity $\Sigma(\Pi)$ for a P-partition Π. Namely, J will be called the Σ-integral of f over I if for each $\varepsilon > 0$ and

$\eta > \eta_0$ (for some fixed η_0), one can find a gauge δ on I such that (1) holds whenever Π is δ-fine and $\Sigma(\Pi) \leq \eta$. The eldest choice for Σ, introduced in [5] together with a definition like above, is

$$\Sigma_0(\Pi) = [\max_{1 \leq j \leq q} \sigma_0(I^j)] / \sigma_0(I) \quad ,$$

where, for a compact interval K of R^n, $\sigma_0(K)$ denotes the ratio between the length of the longest edge of K and that of its smallest edge. The corresponding <u>GP-integral</u> has a Stokes theorem for each differentiable vector field on I but the GP-integrability of f on two abuting intervals does not imply its GP-integrability on their union. Other irregularities were proposed later by Jarnik-Kurzweil-Schwabik [3], Pfeffer [9] and Mkhalfi [7] to overcome this shortcoming and the most interesting, both in power and simplicity, seems to be Mkhalfi's one defined by

$$\Sigma_1(\Pi) = [\max_{1 \leq j \leq q} \sigma_1(x^j, I^j)] / \sigma_0(I) \quad ,$$

where, for a compact interval K of R^n, and $x \in K$, $\sigma_1(x,K)$ denotes the ratio between the farthest and the closest points, with respect to x, lying on the faces of the boundary of K which do not contain x. The corresponding integral, which is called the <u>GM-integral</u>, is meaningful as it is proved that for $\eta \geq 4$ and any gauge δ on I, there exists a δ-fine P-partition Π with $\Sigma_1(\Pi) \leq \eta$. It is obtained by abandoning, in the definition of irregularity of a P-partition, the uniformity requirement with respect to the x^j of Σ_0.

2. An equivalent definition of the GM-integral

One can shed another light to the GM-integral by calling <u>VIP-partition</u> (for Vertex-Indexed-P-partition) of I any P-partition $\Pi = \{(x^1,I^1),\ldots,(x^q,I^q)\}$ of I such that x^j is some vertex of I^j for each $1 \leq j \leq q$. One can show then that f is GM-integrable over I if and only if there is some $J \in R^p$ with the property that for each $\varepsilon > 0$ and $\eta \geq 1$, there exists a gauge δ on I such that (1) holds whenever Π is a δ-fine VIP-partition with $\Sigma_0(\Pi) \leq \eta$. Thus, for n = 2, the GM-integral coincides with the GRC-integral introduced by Buczolich in [1] and Mkhalfi's lemma on the existence of δ-fine P-partitions with $\Sigma_1(\Pi) \leq \eta$ gives easily, for arbitrary n, the existence of δ-fine VIP-partitions with $\Sigma_0(\Pi) \leq \eta$, a result proved in a lengthy way by Buczolich for n = 2 only.

3. An intrinsic divergence of a vector field and Stokes theorem [6]

If $F : \mathcal{J} \to R^p$ is an additive function on the set $\mathcal{J}$ of compact subintervals of $I \subset R^n$, if $x \in I$ and $g(x) \in R^p$, it is natural to call g(x) the σ_1-derivative of G at x, and write

$$g(x) = (\sigma_1\text{-DG})(x) = \sigma_1\text{-}\lim_{J \to x} (G(J)/m(J))$$

if, for each $\varepsilon > 0$ and $\eta \geq 1$, there exists $\gamma > 0$ such that

$$|G(J) - g(x)m(J)| \leq \varepsilon\, m(J)$$

whenever $J \in \mathcal{J}$ is such that $x \in J$, $\sigma_1(x,J) \leq \eta$ and $m(J) \leq \gamma$ (or equivalently diam $J \leq \delta$ or $J \in B[x,\gamma]$). The case where σ_1 is replaced by σ_0 corresponds to the ordinary derivative of an interval function in Saks book [11].

One can then prove the classical property of σ_1-differentiability for the indefinite GM-integral, namely that if f is GM-integrable over I and if

$$F : \mathcal{J} \to R^p, \quad J \mapsto \int_J f$$

is its indefinite integral, then $(\sigma_1\text{-DF})(x)$ exists a.e. on I and is equal to f(x).

Now, given a continuous mapping (vector field for s = 1) $v : I \to (R^s)^n$, $x \mapsto (v_1(x),\ldots,v_n(x))$, let us write

$$\omega_v = \sum_{j=1}^{n} v_j\, dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_n ,$$

$$V : \mathcal{J} \to R^s, \quad J \mapsto \int_{\partial J} \omega_v,$$

$$\sigma_1\text{-div } v(x) = (\sigma_1\text{-DV})(x) .$$

One can show that if v is differentiable at x, then σ_1-div v(x) exists and is equal to

$$\text{div } v(x) = \sum_{j=1}^{n} D_j v_j(x) .$$

Moreover, a Stokes theorem holds for the intrinsic divergence σ_1-div v, namely if σ_1-div v exists on int I and is extended by 0 to

I, then it is GM-integrable over I and

$$\int_I \sigma_1\text{-div } v = V(I).$$

As a consequence, one can get a generalization of a mean value theorem due to Müller-Hermann [8] , namely, under the above conditions, the existence of some $x \in \text{int } I$ such that

$$|V(I)| \leq |\sigma_1\text{-div } v(x)| \; m(I) \quad .$$

4. <u>The σ_1-areal derivative and a characterization of holomorphic functions</u>

The results of Section 3 can be applied to the continuous complex function $h : \Omega \to C$, where $\Omega \subset C$ is open. If $\mathcal{W}$ denotes the set of rectangles contained in Ω and is we define H by

$$H : \mathcal{W} \to C \; , \; J \mapsto (2i)^{-1} \int_{\partial J} h(z)dz \; ,$$

then, according to some pioneering work of Pompeiu [10] , the expression σ_1-$(dh/\alpha)(z)$ defined, where its exists, by

$$\sigma_1\text{-}(dh/d\alpha)(z) = (\sigma_1\text{-}DH)(z) \; ,$$

can be called the <u>σ_1-areal derivative</u> of h at z . One can prove that if σ_1-$(dh/d\alpha)$ exists on $\Omega \setminus \bigcup_{j=1}^{\infty} H_j$, where the H_j are 0- or 1-dimensional hyperplanes parallel to the axes, then h is holomorphic on Ω if and only if σ_1-$(dh/d\alpha) = 0$ a.e. on Ω .

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