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INTEGRALS ON UNBOUNDED INTERVALS AND CONNECTION WITH CONVERGENCE OF DOUBLE SERIES.

1. Preliminaries.

Let $I = [a_1,b_1[\times \ldots \times [a_n,b_n[$ be a left-closed bounded interval of $\mathbb{R}^n$. We define the rate stretching of I, $\sigma_0(I)$, as in [1]

$$\sigma_0(I) = \max_{1\le i\le n} (b_i-a_i)/\min_{1\le i\le n} (b_i-a_i).$$

Now, let π be a P-partition of I, $\pi = \{(x^1,I^1),\ldots,(x^k,I^k)\}$. We consider the Mkhalfi irregularity of π introduced in [2],

$$\Sigma(\pi) = \max_{1\le j\le k} \sigma(x^j,I^j)/\sigma_0(I)$$

where
$$\sigma(x^j,I^j) = \frac{\max\{\text{dist }(x^j,H_i^j)/i=1,\ldots,2n \text{ and } x \notin H_i^j\}}{\min\{\text{dist }(x^j,H_i^j)/i=1,\ldots,2n \text{ and } x \notin H_i^j\}}$$

$H_1^j,\ldots,H_{2n}^j$ are the planes delimiting I^j.

If we have $\Sigma(\pi) \le \eta$ for η fixed, the partition will be η-regular.

Finally, if f is a real function on $\bar{I}$, we say, like in [2], that f is (GM)-integrable on I if there exists $J \in \mathbb{R}$ such that for each $\varepsilon > 0$, for each $\eta \ge 4$ there is a gauge δ on $\bar{I}$ satisfying for each δ-fine η-regular P-partition π of I we have

$$|S(I,f,\pi)-J| \le \varepsilon$$

where $S(I,f,\pi)$ is the Riemann sum associated to π.

2. Definition on three integrals on unbounded intervals

Let I be an unbounded interval of this type

$$I = I_1 \times I_2 \times \ldots \times I_n \quad \text{where } I_i = [a_i,+\infty[\text{ or }]-\infty,b_i[\text{ or }]-\infty,+\infty[\quad \text{for } i=1,\ldots,n,$$

let f be a real function on $\bar{I}$,

We shall say that (1) <u>f is (SGM)-integrable on I</u>
(2) <u>f is (RGM)-integrable on I</u>
(3) <u>f is (RRGM)-integrable on I</u>

if there exists $J \in \mathbb{R}$ such that for each $\varepsilon > 0$, for each $\eta \geq 4$ there is a gauge δ on $\bar{I}$ and a real $r_0 > 0$ satisfying

(1) for every real $r \geq r_0$ and for every δ-fine η-regular P-partition π_r of I_r, one has

$$|S(I_r,f,\pi_r) - J| \leq \varepsilon$$

where $I_r = I \cap ([-r,r[\times \dots \times [-r,r[)$.

(2) for every vector $\vec{r} = (r_1,\dots,r_n)$ with $r \geq r_0$ and for every δ-fine η-regular P-partition $\pi_{\vec{r}}$ of $I_{\vec{r}}$, one has

$$|S(I_{\vec{r}},f,\pi_{\vec{r}}) - J| \leq \varepsilon$$

where $I_{\vec{r}} = I \cap ([-r_1,r_1[\times\dots\times[-r_n,r_n[)$

(3) for every vector $\vec{r} = (r_1,\dots,r_n)$ with $r_i \geq r_0$ and $\sigma_0(I_{\vec{r}}) \leq \eta$, for every σ-fine η-regular P-partition $\pi_{\vec{r}}$ of $I_{\vec{r}}$, one has

$$|S(I_{\vec{r}},f,\pi_{\vec{r}}) - J| \leq \varepsilon.$$

In each case, if J exists, it's unique and we note it (SGM) $\int_I f$, (RGM) $\int_I f$ and (RRGM) $\int_I f$ (resp.).

Moreover denoting by $\mathfrak{S}(I)$, $\mathfrak{R}(I)$ and $\mathfrak{RR}(I)$ the set of the (SGM), (RGM) and (RRGM)- integrable functions on I, we can prove the following inclusions

$$\mathfrak{R}(I) \subset \mathfrak{RR}(I) \subset \mathfrak{S}(I).$$

Among the properties of those integrals, we find, for two of these, a generalisation in multiple dimensions of Hake's theorem.

3. Three kinds of convergence for double series [3].

Let $\sum_{ij} a_{ij}$ be a double real series with partial sums

$$S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$$

One considers three types of convergence for $\sum_{ij} a_{ij}$:

(4) <u>Square convergence</u> : $\sum_{ij} a_{ij}$ S-converges to A ($A \in \mathbb{R}$)

(5) <u>unrestricted Rectangulary convergence</u> : $\sum_{ij} a_{ij}$ R - converges to A

if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}^*$ such that

(4) for every $n \geq n_0$, $|S_{nn} - A| \leq \varepsilon$

(5) for every $m,n \geq n_0$, $|S_{mn} - A| \leq \varepsilon$

(6) <u>Restricted Rectangulary convergence</u> : $\sum_{ij} a_{ij}$ RR - converges to A

if for each $\varepsilon > 0$, for each $\eta > 1$ there is $n_0 \in \mathbb{N}^*$s.th.

(6) for every $m,n \geq n_0$ with $\dfrac{\max(m,n)}{\min(m,n)} = \sigma_0\,([1,m+1[x[1,n+1[) \leq \eta$ one has $|S_{mn} - A| \leq \varepsilon$.

Trivially, the R-convergence of $\sum_{ij} a_{ij}$ implies the RR-convergence of $\sum_{ij} a_{ij}$ which implies it's S-convergence.

4. Relation between integrability and series convergence.

We recall that there exists already a connection between the Perron integral and the convergence of simple series. ([4]) Now for the double series, we can establish a relation between the integrals defined previously and the three types of convergence of $\sum_{ij} a_{ij}$.

Indeed, to each series $\sum_{ij} a_{ij}$, we associate a real function f on the interval $I = [1,+\infty[\times [1,+\infty[$.

$$f(x,y) = a_{ij} \quad \text{if} \quad x \in [i,i+1[,\ y \in [j,j+1[\quad (i\geq 1, j\geq 1).$$

We can now prove the following diagram :

$$
\begin{tikzcd}
\text{R-convergence of } \sum\limits_{ij} a_{ij} \arrow[r] & \text{RR-convergence of } \sum\limits_{ij} a_{ij} \arrow[r] \arrow[d, leftrightarrow] & \text{S-convergence of } \sum\limits_{ij} a_{ij} \\
\text{(RGM)integrability of } f \text{ on } I \arrow[r] \arrow[u] \arrow[ur] & \text{(RRGM)-integrability of } f \arrow[r] \arrow[ul] \arrow[ur] & \text{(SGM)-integrability of } f. \arrow[u] \arrow[ul]
\end{tikzcd}
$$

On the other hand, one can find counter-examples for each implication which doesn't appear into this diagram except for one ; the implication between R-convergence of $\sum\limits_{ij} a_{ij}$ and the (RGM)-integrability of f. This remains presently an open problem.

5. Remark.

One can prove that the S-convergence of $\sum\limits_{ij} a_{ij}$ is equivalent to the (SGM)-integrability of f when $a_{ij} \geq 0$ (but then all the convergences are equivalent to the absolute convergence ; the (SGM)-integral is, in fact the Lebesgue integral and the result is already known) or when $a_{ii} = 0$.

References

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[4] J. MAWHIN, "Introduction à l'Analyse", Cabay (2ème éd.), Louvain-la-Neuve, Belgique (1981) p. 397.