THE PU-INTEGRAL AND ITS PROPERTIES

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0. Introduction

0.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ have a compact support. The PU-integral of f (over $\mathbb{R}^n$) is defined as a limit of Riemannian sums $\sum_i f(t^i) \int \vartheta_i \, dx$ and denoted by $(PU) \int f \, dx$; here $t^i \in \operatorname{supp} f$, $\vartheta_i : \mathbb{R}^n \to [0,1]$ has a compact support and $\{\vartheta_i ; i \in I\}$ is a partition of unity on supp f (I is a finite set). The definition of the PU-integral is given in Section 1. The main features of the PU--integral are summarized in the following points:

0.2. The PU-integral has the usual linear properties. It is an extension of the Lebesgue integral and in general it is nonabsolutely convergent (i.e., the existence of $(PU) \int f \, dx$ does not imply the existence of $(PU) \int |f| \, dx$).

0.3. Let $(PU) \int f \, dx$ exist. If $\phi : \operatorname{supp} f \to \mathbb{R}$ is of class $C^{(1)}$, then $(PU) \int f \phi \, dx$ exists as well and, moreover,

$$\left| (PU) \int f \phi \, dx \right| \leq \text{const.} \ \|\phi\|_1$$

(with the $C^{(1)}$-norm of ϕ).

The measurability of f follows from a suitable form of the Saks--Henstock Lemma, and necessary and sufficient conditions are obtained for a functional $F : C^{(1)} \to \mathbb{R}$ to have a representation in the form

$$F(\phi) = (PU) \int f \phi \, dx .$$

0.4. The usual transformation formula is valid for the PU-integral and $C^{(1)}$-diffeomorphisms; it follows that the PU-integral can be used for integration on manifolds.

<u>0.5.</u> The Stokes' formula holds on a domain with a smooth boundary for (n-1)-forms ω which are differentiable everywhere with the exception of points of a set W, which may be e.g. an r-dimensional manifold; ω has to fulfil some growth conditions near W and the growth conditions depend on the properties of W.

<u>0.6.</u> Let $g : \mathbf{R}^n \to \mathbf{R}$ have a compact support, $1 \leq r \leq n$,

$$g(-x_1,\ldots,-x_r,x_{r+1},\ldots,x_n) = - g(x_1,\ldots,x_r,x_{r+1},\ldots,x_n)$$

for $x \in \mathbf{R}^n$,

$$|g(x_1,\ldots,x_n)| \leq \lambda(x_1^2 + \ldots + x_r^2)\cdot(x_1^2 + \ldots + x_r^2)^{(\ -r)/2}$$

for $0 < x_1^2+\ldots+x_r^2 < 1$, $x_i \in \mathbf{R}$, $i = r+1,\ldots,n$, where $\lambda(\sigma) \searrow 0$ for $\sigma \searrow 0$. For $\sigma > 0$ put $\Omega(\sigma) = \{x \in \mathbf{R}^n;\ x_1^2 + \ldots + x_r^2 \geq \sigma^2\}$ and assume that $(L) \int_{\Omega(\sigma)} g\, dx$ exists for every $\sigma > 0$. Then $(PU) \int g\, dx$ exists and is equal to zero. In particular, if $n = 1$, $r = 1$ and if χ is the characteristic function of the interval $[-1/2,1/2]$, then

$$(PU) \int \chi(x)\cdot\left(x|\ln|x||\right)^{-1} dx = 0 .$$

In general, if $n \geq 1$, if $(PU) \int f\, dx$ exists and if χ is a characteristic function of an interval in $\mathbf{R}^n$, then $(PU) \int f \chi\, dx$ need not exist.

<u>0.7.</u> In case $n = 1$ the PU-integral can be introduced in a simpler way as a limit of integral sums which correspond to certain interval partitions.

In Section 1 the definition of the PU-integral is given and, moreover, it is indicated that partitions required in the definition exist. Sections 2 - 5 contain some comments and details to 0.2 - 0.7.

The PU-integral has been treated in [1] and [2]. However, the present definition differs from the definitions introduced in the above papers; a richer set of integrable functions corresponds to the present definition in comparison with the previous ones, some proofs are simpler and some additional results are obtained.

1. Definition of the PU-integral

1.1. NOTATION AND CONVENTIONS. $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^n$, $n \in \{1,2,\dots\}$. If $t \in \mathbb{R}^n$, $\rho > 0$, then $B(t,\rho) = \{x \in \mathbb{R}^n;\ \|x - t\| < \rho\}$.

Let $Q \subset \mathbb{R}^n$. The set $\Theta = \{(t^i,\vartheta_i);\ i \in I\}$ is called *a system on* Q, if I is a finite set of indices, $t^i \in Q$, $\vartheta_i(t^i) > 0$, $\vartheta_i : \mathbb{R}^n \to [0,1]$ is of class $C^{(1)}$ and has a compact support for $i \in I$ and if $\sum_{i \in I} \vartheta_i(x) \le 1$ for $x \in \mathbb{R}^n$. A system Θ is called *a* PU-*cover of* Q, if

$$\{x;\ \sum_{i \in I} \vartheta_i(x) = 1\} \supset Q .$$

$\int \vartheta_i \, dx$ is written instead of $\int_{\mathbb{R}^n} \vartheta_i(x) \, dx$.

1.2. DEFINITION. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *have a compact support,* $\gamma \in \mathbb{R}$. γ *is called the* PU-*integral of* f *and denoted by* $(PU)\int f \, dx$, *if for every* $\varepsilon > 0$, $K > 1$ *and* $L > 1$ *there exist such functions* $\delta : \operatorname{supp} f \to (0,1]$ *and* $\zeta : \operatorname{supp} f \times (0,1] \to (0,\infty)$ *that*

$$\zeta(t,\sigma) \nearrow \infty , \quad \sigma\zeta(t,\sigma) \searrow 0 \tag{1.1}$$

for every fixed $t \in \operatorname{supp} f$ *and* $\sigma \searrow 0$, *and*

$$|\gamma - \sum_{i \in I} f(t^i) \int \vartheta_i \, dx| \le \varepsilon$$

for every PU-*cover* Θ *of* $\operatorname{supp} f$ *fulfilling*

$$\rho_i < \delta(t^i) \tag{1.2}$$

for $i \in I$ *, where* $\rho_i = \sup \{\|x - t^i\|;\ x \in \operatorname{supp} \vartheta_i\}$,

(1.3) $\quad \|D\vartheta_i(x)\| \le \zeta(t^i,\rho_i)\vartheta_i(t^i)$ *for* $x \in B(t^i,\rho_i/K)$, $i \in I$,

(1.4) $\quad$ *if* $\|D\vartheta_i(x)\| > \zeta(t^i,\rho_i)\vartheta_i(t^i)$, *then*

$$D\vartheta_i(x)(t^i - x) \ge \frac{1}{L}\,\|D\vartheta_i(x)\|\cdot\|t_i - x\|$$

(i.e., the angle of the vectors $D\vartheta_i(x)$ *and* $t^i - x$ *does not exceed* $\arccos 1/L$ *).*

The following theorem makes the above definition meaningful.

<u>1.3. THEOREM</u>. *There exist such* $K_0, L_0 > 1$ *that if* $Q \subset \mathbb{R}^n$ *is compact,* $\delta : Q \to (0,1]$ *, then there exists such a* PU*-cover* Θ *of* Q *that*

(1.2) $\quad \rho_i < \delta(t^i)$ *for* $i \in I$,

(1.3') $\quad D\vartheta_i(x) = 0$ *for* $x \in B(t^i,\rho_i/K_0)$, $i \in I$,

(1.4') $\quad D\vartheta_i(x)(t^i - x) \ge \frac{1}{L_0}\,\|D\vartheta_i(x)\|\cdot\|t^i - x\|$ *for* $x \in \mathbb{R}^n$, $i \in I$.

Theorem 1.3 follows from the next lemma provided regularization is applied to characteristic functions of the sets H_i .

<u>1.4. LEMMA</u>. *Let* $Q \subset \mathbb{R}^n$ *be compact,* $\delta : Q \to (0,1]$ *. Then there exist such compact sets* $H_i \subset \mathbb{R}^n$ *and points* $t^i \in H_i \cap Q$ *for* $i \in I$ *- a finite set - that*

(1.5) $\quad \bigcup_{i\in I} H_i \supset Q$, $\operatorname{Int} H_i \cap \operatorname{Int} H_j = \emptyset$ *for* $i \ne j$,

(1.6) $\quad 0 < \tau_i < \delta(t^i)$, *where* $\tau_i = \sup \{\|x - t^i\|,\ x \in H_i\}$, $i \in I$,

(1.7) $\quad \operatorname{conv}\big(B(t^i,\tau_i/2) \cup \{x\}\big) \subset H_i$ *for* $x \in H_i$, $i \in I$.

2. Basic properties of the PU-integral

2.1. Let $Q \subset \mathbb{R}^n$ be compact. For $f : \mathbb{R}^n \to \mathbb{R}$, $\operatorname{supp} f \subset Q$ define the integral $(PUQ)\int f\,dx$ by the following change in Definition 1.2: use PU-covers of Q instead of PU-covers of $\operatorname{supp} f$ (and assume $\delta : Q \to (0,1]$, $\zeta : Q \times (0,1] \to (0,\infty)$). Then we have:

(2.1) *If* $(PU)\int f\,dx$ *exists, then* $(PUQ)\int f\,dx$ *exists as well and both integrals are equal.*

(2.2) *If* $(PUQ)\int f\,dx$ *exists, then* $(PU)\int f\,dx$ *exists as well and both integrals are equal.*

(2.1) and (2.2) imply that the map $f \mapsto (PU)\int f\,dx$ is linear.

(2.1) follows immediately from the definitions, since it may be assumed without loss of generality that $B(t,\delta(t)) \cap \operatorname{supp} f = \emptyset$ for $t \in Q \setminus \operatorname{supp} f$. The proof of (2.2) is more elaborate.

2.2. In the same way as in [1] it can be proved that $(PU)\int f\,dx$ exists and is equal to $(L)\int f\,dx$ provided the latter integral is finite.

3. Multipliers of PU-integrable functions

3.1. The same scheme as in [2], Section 4 can be used in order to prove the existence of $(PU)\int f\phi\,dx$ for $\phi \in C^{(1)}$, and the inequality from 0.3 can be obtained along similar lines.

3.2. LEMMA (Saks, Henstock): *Let* $(PU)\int f\,dx$ *exist,* $\varepsilon > 0$, $K, L > 1$. *Find* δ, ζ *by Definition 1.2. Then*

$$\left| \sum_{i \in I} \left((PU) \int f\, \vartheta_i\, dx - f(t^i) \int \vartheta_i\, dx \right) \right| \leq \varepsilon$$

for every system Θ *on* supp f *which fulfils* (1.2), (1.3), (1.4).

<u>3.3. THEOREM</u>. *Let* $(PU) \int f\, dx$ *exist. Let* U *be the set of such* $t \in \operatorname{supp} f$ *that for every* $\eta > 0$, $L, K > 1$ *there exist such* $\omega_t \in (0,1]$ *and* $\xi_t : (0,1] \to (0,\infty)$ *fulfilling* (1.1) *that*

$$\left| \int f\, \vartheta\, dx - f(t) \int \vartheta\, dx \right| \leq \eta \int \vartheta\, dx$$

provided $\vartheta : \mathbb{R}^n \to [0,1]$ *is of class* $C^{(1)}$ *with a compact support and*

(1.2″) $\quad 0 < \rho < \omega_t$, *where* $\rho = \sup \{\|x - t\|;\ x \in \operatorname{supp} \vartheta\}$,

(1.3″) $\quad \|D\vartheta(x)\| \leq \xi_t(\rho)\vartheta(t)$ *for* $x \in B(t, \rho/K)$,

(1.4″) $\quad$ *if* $\|D\vartheta(x)\| > \xi_t(\rho)\vartheta(t)$, *then*

$$D\vartheta(x)(t - x) \geq \frac{1}{L} \|D\vartheta(x)\| \cdot \|t - x\| .$$

THEN $m(\operatorname{supp} f \setminus U) = 0$, m *being the Lebesgue measure in* $\mathbb{R}^n$.

<u>3.4. THEOREM</u>. *Let* $(PU) \int f\, dx$ *exist,* $N \subset \operatorname{supp} f$, $m(N) = 0$, $\varepsilon > 0$, $K, L > 1$. *Then there exist such* δ *and* ζ *that*

$$\left| \sum_{i \in I} (PU) \int f\, \vartheta_i\, dx \right| \leq \varepsilon$$

for any system $\Theta = \{(t^i, \vartheta_i);\ i \in I\}$ *on* supp f *fulfilling* (1.2), (1.3), (1.4) *and* $t^i \in N$ *for* $i \in I$.

The PU-integral can be characterized by the properties stated in Theorems 3.3 and 3.4.

<u>3.5. THEOREM</u>. *Let* $F : C^{(1)}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$. *Assume that*

(3.1) $\quad$ supp f *is compact.*

(3.2) F *is algebraically linear,* $F(\vartheta) = 0$ *if* $\vartheta \in C^{(1)}$, $\{x;\ \vartheta(x) \neq 0\} \cap \operatorname{supp} f = 0$.

(3.3) *There exists such a subset* $V \subset \operatorname{supp} f$ *that* $m(\operatorname{supp} f \setminus V) = 0$ *and for every* $t \in V$, $\eta > 0$, $K, L > 1$ *there exist such* $\omega_t \in (0,1]$ *and* $\xi_t : (0,1] \to (0,\infty)$ *fulfilling* (1.1) *that*

$$|F(\vartheta) - f(t) \int \vartheta\, dx| \le \eta \int \vartheta\, dx$$

for any $\vartheta : \mathbf{R}^n \to [0,1]$ *of class* $C^{(1)}$ *which has a compact support and fulfils* (1.2″), (1.3″), (1.4″).

(3.4) *If* $N \subset \operatorname{supp} f$, $m(N) = 0$, $\varepsilon > 0$, $K, L > 1$, *then there exist such* δ *and* ζ *that*

$$\left| \sum_{i \in I} F(\vartheta_i) \right| \le \varepsilon$$

for any system $\Theta = \{(t^i, \vartheta_i);\ i \in I\}$ *fulfilling* (1.2), (1.3), (1.4) *and* $t^i \in N$ *for* $i \in I$.

THEN (PU) $\int f\, dx$ *exists and* $F(\vartheta) = (PU) \int f\, \vartheta\, dx$ *for* $\vartheta \in C^{(1)}(\mathbf{R}^n, \mathbf{R})$.

4. Stokes' theorem

4.1. The following theorem has a crucial role in the proof of Stokes' theorem :

THEOREM. *Let* $g : \mathbf{R}^n \to \mathbf{R}$ *have a compact support and let* W *be the set of such* $x \in \mathbf{R}^n$ *that* $Dg(x)$ *does not exist. Put*

$$f_k(x) = \begin{cases} \dfrac{\partial g}{\partial x_k}(x) & \text{for } x \in \mathbf{R}^n \setminus W , \\ 0 & \text{for } x \in W . \end{cases}$$

THEN (PU) $\int f_k\, dx = 0$ *provided one of the following conditions* *holds:*

(4.1) $n > 1$, $W = \{0\}$, $g(x) = \sigma(\|x\|^{1-n})$ *for* $x \to 0$.

(4.2) $n \geq 1$, $W \subset \{x;\ x_1 = 0\}$, g *is continuous.*

(4.3) $2 \leq p < n$, $W \subset \{x;\ x_1 = x_2 = \dots = x_p = 0\}$, $g(x) = \sigma((x_1^2 + \dots + x_p^2)^{(1-p)/2})$ *uniformly for* $x_1^2 + \dots x_p^2 \to 0$.

(4.4) *There exist such* $\kappa > 0$, $\delta : W \to (0,\infty)$ *that* $\sum_{i \in I} \sigma_i^{n-1} \leq \kappa$ *for every set* $\{(t^i,\sigma_i);\ i \in I\}$, *where* $t^i \in W$, $0 < \sigma_i < \delta(t^i)$, $\|t^i - t^j\| \geq \sigma_i + \sigma_j$ *for* $i \neq j$, $i, j \in I$, I *being a finite set,* g *is continuous.*

(4.5) *For every* $\varepsilon > 0$ *there exists such a function* $\delta : W \to (0,\infty)$ *that* $\sum_{i \in I} \sigma_i^{n-1} \leq \varepsilon$ *for every set* $\{(t^i,\sigma_i);\ i \in I\}$, *where* $t^i \in W$, $0 < \sigma_i < \delta(t^i)$, $\|t^i - t^j\| \geq \sigma_i + \sigma_j$ *for* $i \neq j$, $i, j \in I$, I *being a finite set,* g *is bounded.*

(4.6) $2 \leq p < n$, $W \subset \{x;\ x_1 = x_2 = \dots = x_p = 0\}$ *and for every* $\varepsilon > 0$ *there exists such a function* $\delta : W \to (0,\infty)$ *that* $\sum_{i \in I} \sigma_i^{n-p} \leq \varepsilon$ *for every set* $\{(t^i,\sigma_i);\ i \in I\}$, *where* $t^i \in W$, $0 < \sigma_i < \delta(t^i)$, $\|t^i - t^j\| \geq \sigma_i + \sigma_j$ *for* $i \neq j$, $i, j \in I$, I *being a finite set,* $g(x) = O((x_1^2 + \dots + x_p^2)^{(1-p)/2})$ *uniformly for* $x_1^2 + \dots + x_p^2 \to 0$.

5. An other approach to the PU-integral in case $n=1$

5.1. LEMMA. *For every* $\delta : [0,1] \to (0,1]$ *there exists such a sequence*

(5.1) $$x_0 < t_1 < x_1 < \dots < x_{k-1} < t_k < x_k$$

that

(5.2) $$t_1 = 0\ , \quad t_k = 1\ , \quad t_i - \delta(t_i) < x_{i-1}\ , \quad x_i < t_i + \delta(t_i),$$
$$i = 1,2,\dots,k\ ,$$

(5.3) $$t_i = (x_{i-1} + x_i)/2\ , \quad i = 1,2,\dots,k$$

(cf. [3], [4]).

5.2. DEFINITION. *Let* $f : \mathbb{R} \to \mathbb{R}$, $\operatorname{supp} f \subset [0,1]$, $\gamma \in \mathbb{R}$. γ *is called the* AS_1*-integral of* f *and denoted by* $(AS_1) \int f\,dx$, *if for every* $\varepsilon > 0$ *and* $K > 0$ *there exists such a* $\delta : [0,1] \to (0,1]$ *that*

$$\left| \gamma - \sum_{i=1}^{k} f(t_i)(x_i - x_{i-1}) \right| \le \varepsilon$$

for every sequence (5.1) *fulfilling* (5.2) *and*

(5.4) $$x_i - t_i \le (1 + K)(t_i - x_{i-1})\ ,$$
$$t_i - x_{i-1} \le (1 + K)(x_i - t_i)\ .$$

The definition was introduced in [3]; Lemma 5.1 makes it meaningful.

5.3. THEOREM. *Let* $f : \mathbb{R} \to \mathbb{R}$, $\operatorname{supp} f \subset [0,1]$. *If one of the integrals in*

(5.5) $$(PU) \int f\,dx = (AS_1) \int f\,dx$$

exists, then the other exists as well and (5.5) *holds.*

5.4. NOTE. The AS_1-integral is an extension of the Perron integral.

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