Piotr Mikusiński, Department of Mathematics, University of Central Florida, Orlando, FL 32816
Krzysztof Ostaszewski, Department of Mathematics, University of Louisville, Louisville, KY 40292[1]

Embedding Henstock integrable functions into the space of Schwartz distributions

Introduction. In this work we show how the space of Henstock integrable functions of several variables can be viewed as a subspace of a certain space of Schwartz distributions. Also, we show how probability distribution functions are multipliers for Henstock integrable functions, and generate continuous linear functionals on the space of Henstock integrable functions.

1.1. Definition. Let $I_0 \subset \mathbf{R}^m$ be the unit cube in the m-dimensional Euclidean space. A function $f : I_0 \to \mathbf{R}$ will be termed *Henstock integrable*, with

$$\int\int\cdots\int_{I_0} f(x_1, x_2, \ldots, x_m)dx_1dx_2\ldots dx_m \tag{1}$$

written for the value of the integral, if for every $\varepsilon > 0$ there exists a positive function $\delta : I_0 \to \mathbf{R}$ (usually called a *gauge*) such that whenever

$$\pi = \left\{ \left((x_1^i, x_2^i, \ldots, x_m^i), I_i\right) : i = 1, 2, \ldots, n \right\} \tag{2}$$

is a partition of I_0, consisting of pairs of points in I_0 and nonoverlapping subintervals of I_0 whose union is the whole I_0, and such that for every $i = 1, 2, \ldots, n$, $(x_1^i, x_2^i, \ldots, x_m^i) \in I_i$ and I_i is contained in the ball centered at $(x_1^i, x_2^i, \ldots, x_m^i)$ of radius $\delta(x_1^i, x_2^i, \ldots, x_m^i)$, we have

$$\left|\sum_{i=1}^{n} f(x_1^i, x_2^i, \ldots, x_m^i)\lambda(I_i) - \int\int\cdots\int_{I_0} f(x_1, x_2, \ldots, x_m)dx_1dx_2\ldots dx_m\right| < \varepsilon; \tag{3}$$

here $\lambda(I)$ stands for the m-dimensional volume of an interval $I \subset \mathbf{R}^m$. Quite often we will simply write

$$\int_{I_0} f\, d\lambda \tag{4}$$

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for the Henstock integral of f over I_0.

A partition π as in (2), satisfying conditions listed between (2) and (3) will be called *δ-fine.*

We will also denote by $\tilde{f}$ the indefinite Henstock integral of a function f, i.e.,

$$\tilde{f}(x_1, x_2, \ldots, x_m) = \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_m} f(t_1, t_2, \ldots, t_m)\, dt_1 dt_2 \ldots dt_m =$$

$$\int \int \cdots \int_{I_0} f(t_1, t_2, \ldots, t_m) \chi_{[0,x_1]\times[0,x_2]\times[0,x_m]}\, dt_1 dt_2 \ldots dt_m, \tag{5}$$

where χ_E denotes the characteristic functions of a set $E \subset \mathbf{R}$.

1.2. The class of Henstock integrable functions on I_0 will be denoted by $\mathcal{H}$. It is a linear topological space. In [8] and [9] it is shown that the space equipped with the Alexiewicz norm is barrelled, but it is not a Banach space. [6] and [8] discuss the dual of the space. The work in [8] is done in the two-dimensional case, but easily extends to the multidimensional one. [6] considers the dual of $\mathcal{H}$ for functions of one variable.

Our intention is to describe the completion of the space and to further discuss its dual.

Let us note that every Henstock integrable function $f: I_0 \to \mathbf{R}$ is a Schwartz distribution (see [4], section 2.12).

1.3. Definition. Denote by $\mathcal{F}$ the space of all *distributions of order m* with support in I_0, i.e., $f \in \mathcal{F}$ if there exists a continuous function $F: \mathbf{R}^m \to \mathbf{R}$ such that

$$F(x_1, x_2, \ldots, x_m) = 0 \text{ if } \min\{x_1, x_2, \ldots, x_m\} \le 0, \tag{6}$$

$$F(x_1, x_2, \ldots, x_i, \ldots, x_m) = F(x_1, x_2, \ldots, 1, \ldots, x_m) \text{ if } x_i \ge 1 \text{ for } i = 1, 2 \ldots, m, \tag{7}$$

$$f = \frac{\partial^m F}{\partial x_1 \partial x_2 \ldots \partial x_m}, \tag{8}$$

where the derivatives are understood in the distributional sense.

For $f \in \mathcal{F}$ and F as in (6), (7), and (8) define

$$\int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_m} f(t_1, t_2, \ldots, t_m)\, dt_1 dt_2 \ldots dt_m = F(x_1, x_2, \ldots, x_m) \tag{9}$$

for $(x_1, x_2, \ldots, x_m) \in I_0$. Note that, for every $f \in \mathcal{F}$ there exists exactly one function F satisfying (6), (7), and (8). Thus the integral (9) is uniquely defined. Moreover

$$\|f\| = \sup_{(x_1,x_2,\ldots,x_m)\in I_0} |F(x_1,x_2,\ldots,x_m)| \tag{10}$$

is a norm on $\mathcal{F}$. We will call it the *Alexiewicz norm*, as it is the same as the Alexiewicz norm introduced in [8] on the space of Henstock integrable function.

1.4. Proposition. *$\mathcal{F}$ is complete.*

1.5. Observation. $\mathcal{H} \subset \mathcal{F}$.

1.6. Theorem. *$\mathcal{F}$ is the completion of $\mathcal{H}$.*

1.7. Remark. In the one-dimensional case it is known that every Henstock integrable function is almost everywhere a derivative of its indefinite integral. This implies that in that case, $\mathcal{H}$ is of the first category in $\mathcal{F}$. An easy example of an element of $\mathcal{F}$ which is not in $\mathcal{H}$ is in that case a distributional derivative of a nowhere differentiable continuous function.

2.1. We will turn now to our discussion of the dual of the space $\mathcal{H}$. We have the following, as presented in [6] and [8]:

In the one-dimensional case T is a continuous linear functional on $\mathcal{H}$ if and only if either of the following holds (all integrals used below are Henstock integrals):
(a) There exists a finite signed Borel measure μ_T on $(0,1]$ such that

$$T(f) = \int_0^1 \tilde{f}(t)\, d\mu_T(t), \tag{11}$$

where, as usually

$$\tilde{f}(x) = \int_0^x f(t)\, dt. \tag{12}$$

(b) There exists a function $g_T : [0,1] \to \mathbf{R}$ of essentially bounded variation such that

$$T(f) = \int_0^1 f(t) g_T(t)\, dt. \tag{13}$$

Being of essentially bounded variation is equivalent to having a signed finite Borel measure as a distributional derivative. If μ_g stands for that distributional derivative then integration by parts yields

$$\int_0^1 \tilde{f}(t)\, d\mu_T(t) = \int_0^1 \tilde{f}(t)\, d\mu_g(t) + \tilde{f}(1)\mu_g((0,1]). \tag{14}$$

Notice that the expression $\tilde{f}(1)\mu_g((0,1])$ is itself a continuous linear functional of f.

As observed in [8] the description (a) easily extends to the multidimensional case. However, (b) uses the class of multipliers for the Henstock integrable functions (i.e., functions which multiplied by a Henstock integrable function produce a Henstock integrable function), which is not known in the multidimensional case.

To simplify our considerations let us restict ourselves to the two-dimensional case, with $I_0 = [0,1] \times [0,1]$. This does not effect generality of the results.

2.2. Definition. A function $g : I_0 \to \mathbf{R}$ is of *strongly bounded variation* (see [4]) if for every $x \in [0,1]$, $g(x,\cdot)$ is of bounded variation, for every $y \in [0,1]$, $g(\cdot, y)$ is of bounded variation, and

$$\sup \sum_{i=1}^{n} |g(a_i, c_i) - g(a_i, d_i) - g(b_i, c_i) + g(b_i, d_i)| < +\infty, \tag{15}$$

where the least upper bound is taken over all partitions of I_0 into a finite collection of nonoverlapping nondegenerate closed intervals $[a_i, b_i] \times [c_i, d_i]$, $i = 1, 2, 3, \ldots n$.

Let us note that [4] contains the definition of a function of strongly bounded variation in the general multidimensional case.

2.3. Theorem. (Kurzweil [4]) *Every function of strongly bounded variation is a multiplier for Henstock integrable functions.*

2.4. It is not known whether the above is a complete characterization of multipliers. Our intention is to points out a specific subclass of the class of functions of strongly bounded variation.

2.5. Definition. Let $\mathcal{D}$ stand for the class of two-dimensional distribution functions of finite signed Borel measures on $(0,1] \times (0,1]$. For example, if μ is a positive measure then $g_\mu \in \mathcal{D}$ corresponding to it is

$$g_\mu(x,y) = \mu\big((0,x] \times (0,y]\big). \tag{16}$$

The value of $g_\mu(x,y)$ for $x = 0$ or $y = 0$ is unessential to us, we will assume it to be zero.

In general, for a signed finite Borel measure μ on $(0,1] \times (0,1]$ we will denote its distribution function by g_μ.

Also, we will denote by $\mathcal{M}$ the class of finite signed Borel measures on $(0,1] \times (0,1]$. $\mathcal{M}^+$ will denote the class of positive measures in $\mathcal{M}$.

2.6. Proposition. *The elements of $\mathcal{D}$ are of strongly bounded variation.*

2.7. Corollary. *A distribution function of a finite signed Borel measure is a multiplier for Henstock integrable functions.*

2.8. Corollary. *If $g : I_0 \to \mathrm{R}$ is equivalent to a distribution function of a finite signed Borel measure then g is a multiplier for Henstock integrable functions.*

2.9. Definition. Let $\mathcal{C}_0$ denote the class of all continuous $F : I_0 \to \mathrm{R}$ such that F is continuous and $F(x,y) = 0$ whenever $x = 0$ or $y = 0$.

2.10. Proposition. *Let $f \in \mathcal{H}$ and $\mu \in \mathcal{M}$. Then*

$$\begin{aligned}
&\int\int_{I_0} f(x,y) g_\mu(x,y)\, dx dy = \\
&\tilde{f}(1,1) g_\mu(1,1) - \int_0^1 \tilde{f}(t,1) d g_\mu(t,1) - \int_0^1 \tilde{f}(1,t) d g_\mu(1,t) + \int\int_{I_0} \tilde{f}\, d\mu.
\end{aligned} \tag{16}$$

2.11. Remark. For $\mu \in \mathcal{M}$ the expression

$$\int\int_{I_0} f(x,y) g_\mu(x,y) dx dy \tag{17}$$

is a continuous linear functional on $\mathcal{H}$. We do not know, however, if (17) gives the general form of a continuous linear functional on $\mathcal{H}$. As we stated in 2.1, [8] shows that the general form of a continuous linear functional on $\mathcal{H}$ is

$$\int\int_{I_0} \tilde{f}\, d\mu, \tag{18}$$

where $\mu \in \mathcal{M}$. Proposition 2.10 suggests the hypothesis that (58) is in fact another general form of a continuous linear functional on $\mathcal{H}$. We were not able to either prove or disprove it.

Also the following two problems are very natural.

2.12. Problem. Given a function $g : I_0 \to \mathrm{R}$ of strongly bounded variation, is there a $\mu \in \mathcal{M}$ such that g is equivalent to g_μ?

2.13. Problem. Given a multiplier g for Henstock integrable functions, is there a $\mu \in \mathcal{M}$ such that g is equivalent to g_μ?

References:

1. S.I. Ahmed, and W. F. Pfeffer, A Riemann integral in a locally compact Hausdorff space, *J. Austral. Math. Soc.*, **41**A (1986), 115-137.
2. R. Henstock, *Theory of integration*, Butterworths, 1963.
3. R. Henstock, Integration, variation, and differentiation in division spaces, *Proc. Royal Irish Acad.*, **78**A (1978), 69-85.
4. J. Kurzweil, On multiplication of Perron-integrable functions, *Czech. Math. J.*, **23**(98) (1973), 542-566.
5. J. Kurzweil, *Nichtabsolut konvergente Integrale*, Teubner Texte zür Mathematik, No. 26, Leipzig, 1980.
6. K. Ostaszewski, A topology for the spaces of Denjoy integrable functions, Proceedings of the Sixth Summer Real Analysis Symposium, *Real Analysis Exchange*, **9**(1) (1983-84), 79-85.
7. K. Ostaszewski, *Henstock Integration in the Plane*, Memoirs of the Amer. Math. Soc., (63)**353**, September 1986.
8. K. Ostaszewski, The space of Henstock integrable functions of two variables, *Internat. J. Math. and Math. Sci.*, (11)**1** (1988), 15-22.
9. B.S. Thomson, Spaces of conditionally integrable functions, *J. London Math. Soc.*, (2)**2** (1970), 358-360.