INTEGRATION OVER FUNCTION SPACES

by Ralph Henstock

The origins of integration over function spaces are in papers of Einstein and Smoluchowski. Two integrals vital for theoretical physicists and theoretical chemists are given by Wiener and Feynman, though the latter never gave any mathematical formulation of that integral, and there is not even one integral sign in the 1947 paper. We consider paths $x(b)$ parametrised by a real variable b, such as the time, lying in an interval $[a,+\infty)$. The generalized intervals I we use are the set of those paths that, when $b = b_j$, lie in the range

$$u(b_j)\leqslant x(b_j)<v(b_j),\ u(b_j)<v(b_j)\ (1\leqslant j\leqslant n),\ a<b_1<\ldots<b_n\ .$$

A possible function of such I, is the integral

$$P(I) \equiv \int_{u(b_1)}^{v(b_1)} \cdots \int_{u(b_n)}^{v(b_n)} p(x_1,\ldots,x_n;C)\,dx_1\ldots dx_n\ (C = (b_1,\ldots,b_n)),$$

$$p(x_1,\ldots,x_n;C) = q(x_1;b_1-a|x_o)q(x_2;b_2-b_1|x_1)\ldots q(x_n;b_n-b_{n-1}|x_{n-1}),$$

with q obeying a consistency condition of Smoluchowski, Chapman, and Kolmogorov, namely, that for each fixed s in $0<s<t$,

$$q(y;t|x) = \int_{-\infty}^{\infty} q(z;s|x)q(y;t-s|z)\,dz.$$

For example, the path of a free spherical Brownian particle starting at time $b = a$ from $x = 0$ has an attached probability measure constructed by taking $x_o = 0$ and

$$q(y;t|x) = g(y-x;4Dt)\ \text{with}\ g(x;t) = (\pi t)^{-\frac{1}{2}}\exp(-x^2/t).$$

D is the diffusion constant that depends on the viscosity and temperature of the medium and the radius of the Brownian particle.

When $D = \frac{1}{2}$, $P(I)$ is the Wiener measure $W(I)$ of I. Feynman used complex-valued q , one of them having $D = i/4$.

The intervals at $b = b_1, b_2, \ldots, b_n$ are just like the edges of an n-dimensional rectangle and the whole collection of paths is the definition of the Cartesian product of intervals $(-\infty,\infty)$, one interval for each b. Thus in the division space setup we consider a Cartesian product of one-dimensional division spaces $((-\infty,\infty),\mathcal{J},\mathbf{A})$ for each $b>a$, in the usual notation, where $\mathcal{J}$ is the collection of all intervals $[u,v)$ on $(-\infty,\infty)$ and $\mathbf{A}$ is the generalized Riemann setup that uses a function $\delta>0$, together with the usual arrangements for $-\infty$ and $+\infty$. Actually we use a general Cartesian product of arbitrary division spaces, whether one-dimensional or not. To prove that suitable divisions can be found, we prove compactness for a suitable topology on each separate division space and then use Tychonoff's theorem to show the compactness of the whole space. The topology comes from an unexpected source, the division space setup itself. An *elementary set* E is an interval or a finite union of disjoint intervals, and a *division* of E is a finite number of interval-points (I,t) such that the I are disjoint with union E. We now focus our attention on the points t, and the set $E^*(\delta)$ or $E^*(U)$ of all t that are in the (I,t) for all δ-fine divisions of E, or for the corresponding divisions of E formed from members of a family U of (I,t) that lies in A. The intersection E^* of all such sets $E^*(U)$ is called a *star-set*, and it behaves very much like a closed set. In fact we construct a topology, the *intrinsic topology*, from the complements $\backslash E^*$. If P_1 and P_2 are two disjoint elementary sets contained in E

(called *partial sets*) then $P_1 \cup P_2$ is also a partial set and $(P_1 \cup P_2)^* = P_1^* \cup P_2^*$. Thus the intrinsic topology is the empty set with the complements $\backslash P^*$. We assume that, given elementary sets $P \subset E$, there is a $U_P \in A$ such that if $T \nsubseteq P^*$ and $(I,t) \in U$ then $I^* \subseteq E^* \backslash P^*$. This is reasonable since $t \in E^* \backslash P^*$ anyway. We also assume that the space is fully decomposable. Then E^* *is compact in the intrinsic topology*. The easy proof is as follows. We need only consider covers of E^* that are families of $\backslash P^*$, so that each $t \in E^*$ lies in one of the $\backslash P^*$, say, $\backslash P(t)^*$. By full decomposability we can stick together parts of the $U_{P(t)}$ to form another $U \in A$ dividing E, say with a division D. For $(I,t) \in D$ we have $(I,t) \in U_{P(t)}$, $t \in \backslash P(t)^*$, so $I^* \subseteq E^* \backslash P(t)^*$, and $E^* = \cup_D I^* \subseteq \cup_D E^* \backslash P(t)^*$. As only a finite number of (I,t) lies in D we cover E^* by a finite number of $\backslash P(t)^*$ from the topology, and E^* is compact. We now proceed via Tychonoff to the Cartesian product space, which is therefore compact. So every family of closed sets with the finite intersection property (every finite number of sets from the collection has a common point) has a common point. This is used to show that every elementary set in the Cartesian product space has a division, by using continued bisection or a similar construction. Note that the preceding remarks need much more technical detail if the space does not have the additive property.

In countable Cartesian product spaces (sequence spaces) a result of Jessen holds, so that the integral is the limit of the corresponding integral over the first n terms of the sequences, *i.e.* integration in n dimensions, plus the limit of n. The integral over an uncountable Cartesian product also can often be proved to be the limit of integrals over spaces of finite dimensions, which is a great help in evaluation.