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INFINITE-DIMENSIONAL GENERALISED RIEMANN INTEGRALS

In one dimension, we take intervals I of the form

$$]-\infty, a[,\ [u, v[,\ [b, \infty[$$

and define |I| as

$$a,\ v - u,\ b$$

respectively. The pair (I, x) are associated if x is

$$-\infty,\ u \text{ or } v,\ +\infty$$

respectively. A gauge is a positive function $\delta(x)$ defined for $x \in R^* = R \cup \{-\infty, \infty\}$.
If I, x are associated then (I, x) or, simply, I, is δ-fine if |I|

$$< -1/\delta(x),\ < \delta(x),\ > 1/\delta(x)$$

respectively.

In n dimensions, an interval I is $I_1 x \ \ldots \ x\ I_n$, where each I_j is an interval of R.
|I| is the product of the $|I_j|$. If $x = (x_1, ..., x_n)$, $x_j \in R^*$, then (I, x) is an associated couple if (I_j, x_j) are associated in R, $1 \le j \le n$. Given $\delta(x) > 0$, I and (I, x) are δ-fine if I_j is δ-fine, $1 \le j \le n$. A partition of R^n is a finite collection of disjoint intervals I whose union is R^n. A division of R^n is a finite collection of associated (I, x) such that the intervals I form a partition of R^n. A division $\mathfrak{E}$ of R^n is δ-fine if each $(I, x) \in \mathfrak{E}$ is δ-fine.

Given a point function h(x), the Riemann sum is denoted by

$$(\mathfrak{E})\Sigma h(x)|I|,$$

or simply $\Sigma h(x)|I|$, where we take h(x) equal to zero by definition if x is at infinity.

Similarly, if h(I, x) is a function of associated (I, x), the Riemann sum is $\Sigma h(I, x)$, making a similar provision whenever x has an infinite component.

For the infinite dimensional case, let $]\tau', \tau[$ be an interval of R, and let $T = R^{]\tau', \tau[}$. Thus T is the set

$$\{x : t \to x(t), t \in]\tau', \tau[, x(t) \in R\}.$$

T* is T with points at infinity adjoined, i.e. replace R by R* in previous definition.
Let $N = \{t_1, ..., t_n\}$, $x_j = x(t_j)$ for $1 \le j \le n$, $x(N) = (x_1, ..., x_n)$,
$I = I[N] = \{x : x(t_j) \in I_j, 1 \le j \le n\}$, $I(N) = I_1 \times \ \ldots \ \times I_n$, $|I| = |I[N]| = |I_1|. ... .|I_n|$.
(I[N], x) are associated in T if (I(N), x(N)) are associated in R^n. The definitions of partition and division of T follow those given for R^n.

A gauge in T is a device which restricts or reduces the volume $|I| = |I[N]|$ by
(i) increasing the set of restricted dimensions N, and
(ii) reducing the lengths of the restricted edges I_j, $t_j \in N$.

Let A be a countable subset of $]\tau', \tau[$. For each $x \in T^*$, let L(x) be a finite subset of A, and, for each $N \supseteq L(x)$, let $\delta(x(N)) > 0$ be defined. Then a gauge γ is defined as

$$\{(A, L(x), \delta(x(N)) : x \in T\}.$$

(R.Henstock has given a proof, not yet published, that the countable set A can be omitted from this definition so that, for each $x \in T^*$, L(x) is an arbitrary finite subset of $]\tau', \tau[$.)
If I[N], x are associated then (I[N], x), or I, is γ-fine if $N \supseteq L(x)$ and I(N) is δ-fine. A division of T is γ-fine if each member of the division is γ-fine.

A functional h(I, x, N) of associated triples (I, x, N) will be zero by definition if x(N) has an infinite component. h is integrable in T with $\int h = \alpha$ if, given $\varepsilon > 0$, there exists a gauge γ so that for every γ-fine division of T,

$$|\Sigma h(I, x, N) - \alpha| < \varepsilon.$$

Let c denote a complex number $a + ib$, and let $w_c(x, N)$ be defined as

$$\exp\left(c\sum_{j=1}^{n}((x_j - x_{j-1})^2/(t_j - t_{j-1}))\right)\prod_{j=1}^{n}(-\pi(t_j - t_{j-1})/c)^{-1/2}.$$

Let $w_c(I, x, N) = w_c(x, N)|I[N]|$.

The latter functional is integrable in T, ((1), p.84), provided $a \leq 0$, $b \geq 0$, $c \neq 0$.

The case $c = -1/2$ gives the Wiener integrator, while $c = i/2$ gives the Feynman integrator.

Let $p > 0$, and let $C = \{ x : |x(t) - x(t')| < |t - t'|^p$ as $t \to t'$, $t, t' \in [\tau', \tau]\}$. Let $D = T \setminus C$.

THEOREM : If $p < 1/2$, $b > 0$, then $\int \chi(D, x) w_{ib}(I, x, N)$ exists and equals zero.

This implies that the Feynman integrals are path integrals, as the discontinuous functions are a null set relative to the Feynman integrator.

REFERENCE

1. Muldowney, P., A General Theory of Integration in Function Spaces, Pitman Research Notes in Mathematics, 1987.