Real Analysis Exchange Vol. 12, Number 2 (1986-87)

Janusz Woś, Institute of Mathematics, Technical University,
Wybrzeze Wyspiańskiego 27, 50-370 Wrocław, Poland

ON APPROXIMATE DIFFERENTIABILITY AND BILATERAL KNOT POINTS

It is well known that for a measurable function f defined on the interval $[0,1]$ we have $D_-f = ADf = D^+f$ a.e. in $\{D^+f < +\infty\}$. (See [1], Theorem 2.) At the same time there are trivial examples of measurable functions f for which the approximate derivative ADf exists a.e., but almost every point $x \in [0,1]$ is a bilateral knot point, i.e. $D^+f(x) = D^-f(x) = +\infty$ and $D_+f(x) = D_-f(x) = -\infty$. In this note, answering a problem of K. M. Garg ([2], 10.3 Problem), we construct a continuous function with these properties.

Let us consider three sequences $\{c_n\}$, $\{d_n\}$, and $\{r_n\}$ of positive real numbers such that $\{d_n\}$ is decreasing, $c_n < d_n/2$, and

(1) $\lim_{n\to\infty} r_n d_n^{-1} = \infty$,

(2) $\sum_{n=1}^{\infty} c_n d_n^{-1} < \infty$,

(3) $\sum_{n=1}^{N-1} r_n c_n^{-1} \leq r_N/4d_N$ for each $N \geq 2$,

(4) $\sum_{n=N+1}^{\infty} r_n \leq r_N/4$ for each $N \geq 1$.

In order to provide some examples of such sequences let us take a sequence $\{u_n\}$ of numbers from the interval $(0,1/2)$ satisfying $\sum_{n=1}^{\infty} u_n < \infty$, and let $\{w_n\}$ be an arbitrary sequence of positive real numbers which decreases to zero and satisfies the inequality $\sum_{n=1}^{N-1} (u_n w_n)^{-1} \leq (4w_N)^{-1}$ for every $N \geq 2$. Next, for arbitrary fixed number $p \geq 5$, put

$$r_n = p^{-n} , \quad d_n = w_n r_n , \quad c_n = u_n d_n \quad (n \geq 1) .$$

It is easy to see that these three sequences have the desired properties.

Now, let $k_n = [d_n^{-1}]$ be the integer part of d_n^{-1}. For an arbitrary positive integer n and for an arbitrary $x \in [0,1]$ we put

$$f_n(x) = \begin{cases} (-1)^{i+1} r_n(1 + c_n^{-1}(x - id_n)) & \text{for } x \in [id_n - c_n, id_n], \\ & i = 1,2,\dots,k_n, \\ (-1)^{i+1} r_n(1 - c_n^{-1}(x - id_n)) & \text{for } x \in [id_n, id_n + c_n], \\ & i = 1,2,\dots,k_n, \\ 0 \quad \text{otherwise.} & \end{cases}$$

Clearly, each f_n is continuous and piecewise linear. Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Since $|f_n| \leq r_n$, we have $\sum_{n=1}^{\infty} |f_n(x)| \leq \sum_{n=1}^{\infty} r_n < \infty$ by (4). Consequently, f is continuous.

Writing

$$f = \sum_{n=1}^{N-1} f_n + \sum_{n=N}^{\infty} f_n = g_N + b_N ,$$

we see that g_N is differentiable a.e. (Actually, being piecewise linear, it is differentiable outside a finite number of points.) and b_N vanishes on the set

$$A_N = [0,1] \setminus \bigcup_{n=N}^{\infty} \operatorname{supp} f_n .$$

However, by the definition of f_n and by (2), we have

$$\sum_{n=1}^{\infty} |\operatorname{supp} f_n| \leq 2 \sum_{n=1}^{\infty} c_n k_n \leq 2 \sum_{n=1}^{\infty} c_n d_n^{-1} < \infty ,$$

which yields

(5) $\qquad \lim_{N\to\infty} |A_N| = 1 .$

Consequently, f is approximately differentiable a.e. by the Lebesgue density theorem.

Now, let $B_N = A_N \setminus ([0,2d_N] \cup [1 - 2d_N,1])$ and let $N \geq 1$ be fixed. Take arbitrary $x \in B_N$. By the definition of f_N there exist positive integers l_1, l_2, l_3, l_4 such that

(6) $\qquad l_1 d_N < x , \quad l_2 d_N < x , \quad l_3 d_N > x , \quad l_4 d_N > x ,$

(7) $\quad |l_j d_N - x| < 2d_N \quad (j = 1,2,3,4),$

and

(8) $\quad f_N(l_1 d_N) = f_N(l_3 d_N) = r_N, \quad f_N(l_2 d_N) = f_N(l_4 d_N) = -r_N$.

Since $x \in A_N$, we have $f_n(x) = 0$ for $n \geq N$. Consequently,

$$\frac{f(l_j d_N)-f(x)}{l_j d_N - x} = \Sigma_{n=1}^{N-1} \frac{f_n(l_j d_N)-f_n(x)}{l_j d_N - x} + \Sigma_{n=N}^{\infty} \frac{f_n(l_j d_N)}{l_j d_N - x}$$

$$= D_1 + D_2 .$$

By definition of f_n and by (3) we obtain

(9) $$|D_1| \leq \Sigma_{n=1}^{N-1} \sup_{x \neq y} \left| \frac{f_n(y) - f_n(x)}{y - x} \right|$$

$$\leq \Sigma_{n=1}^{N-1} r_n c_n^{-1} \leq r_N/4d_N .$$

Moreover, by (4) we have

(10) $$|\Sigma_{n=N+1}^{\infty} f_n(l_j d_N)| \leq \Sigma_{n=N+1}^{\infty} r_n \leq r_N/4 .$$

Now, let us consider the case when $j = 1$. Then $f_N(l_1 d_N) = r_N$, by (8). Using (10), (6), and (7) we obtain

$$D_2 \leq 3r_N/4(l_1 d_N - x) \leq -3r_N/8d_N$$

which, together with (9), yields

(11) $$D_1 + D_2 \leq -r_N/8d_N .$$

Since d_N decreases to zero and the sets A_N are increasing, the sets B_N are increasing and $|\cup_{N=1}^{\infty} B_N| = 1$, by (5). This, together with (11) and (1), implies that the lower left Dini derivative D_-f is equal to $-\infty$ a.e.

Similarly, considering the cases when $j = 2,3,4$ we obtain that $D^-f = +\infty$, $D^+f = +\infty$, $D_+f = -\infty$ a.e., respectively.

We end this note with the following

PROBLEM. Let $Q = \{q_1,q_2,\ldots\}$ be a fixed sequence of positive real numbers decreasing to zero. Does there exist on the interval $[0,1]$ a continuous function f such that (a) f is approximately differentiable a.e., (b) almost every point is a bilateral knot point, and (c) the sequential path derivative $D_Q f(x) = \lim_{n\to\infty} (f(x + q_n) - f(x))/q_n$ exists a.e.?

REFERENCES

[1] J.C. Burkill, U.S. Haslam-Jones, The derivates and approximate derivates of measurable functions, Proc. London Math. Soc. 32 (1930), 346-355.

[2] K.M. Garg, Some new notions of derivative, Memoirs Amer. Math. Soc. (to appear).

Received August 25, 1986