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SYMMETRIC FUNCTIONS WHOSE SET OF POINTS OF DISCONTINUITY IS UNCOUNTABLE

An extended real function f(x) is said to be symmetric if
$$\lim_{h\to 0} f(x+h)+f(x-h)-2f(x)=0,$$
and symmetrically continuous if
$$\lim_{h\to 0} f(x+h)-f(x-h)=0.$$
The investigation of the set of points of discontinuity for symmetrically continuous and symmetric functions dates back to 1935 when F. Hausdorff [2] asked whether this set could be uncountable. An affirmative answer to this question for symmetrically continuous functions was given by David Preiss in 1971 using a certain type of convergent Fourier series [6], but he left the question open for symmetric functions [4]. In this paper we construct a measurable symmetric function f(x) discontinuous on an uncountable set. Furthermore, the function $g(x)=|f(x)|$ is another example of a symmetrically continuous function discontinuous on an uncountable set. We also construct such a function, which has c points of discontinuity in every interval. We note that [1], [5] a construction like this must fail for a measurable smooth function because the set of points of discontinuity of such functions has been shown to be scattered, and therefore countable.

Example: There exists a measurable symmetric function, whose set of points of discontinuity is uncountable.

Construction: Since the construction is complicated, it will pay to have a rough idea of how it proceeds. We begin with the function $H(x)$ linear between any two consecutive of the points (0,0), (1/4,1), (3/4,-1), and (1,0), and $H(x)=0$ for x outside [0,1].

Heuristically speaking, disjoint copies of the graph of jagged functions of the form $pH(kx-r)$ will be successively placed above subintervals of [0,1]. Initially, a function f_1 is formed by placing eight copies with successive heights 1/4, 3/4, 1, 1, 1, 1, 3/4, 1/4 in eight subintervals. On seven smaller intervals between the eight, $f_1(x)=0$. Then in each of the seven intervals, the function f_2 is formed by placing twenty disjoint copies with their heights increasing to the average height of f_1 on the two adjacent intervals, remaining at this average for four copies and then decreasing. Thus the graph of f_2 consists of 140 jagged functions in seven groups of twenty each. Nineteen smaller intervals are retained between each of the twenty, where $f_2(x)=0$, and the process is repeated in these intervals. The sum of the resulting sequence of functions is $f(x)$.

We now begin the construction.

Successive partitions of the unit interval:

The unit interval [0,1] is divided into $N_1=15$ intervals denoted by $I(0), I(1), \ldots, I(14)$ so that the length of even indexed intervals is d_1, and the length of odd indexed intervals is r_1, and $d_1=(2^1+1)r_1=3r_1$. The unit interval [0,1] is divided into $2^{3.1} + 2^{2.1+1} + 2^{1+2} + 7=31$ subintervals of equal length r_1.

Let $c_1=1/_{31}$ and note that $r_1=c_1$. Stage 2 intervals are denoted by $I(n_1,n_2)$, $0\leqslant n_i < N_i$, $N_i=2.2^{2i}+7$, $i=1,2$ and n_1 odd. They are formed as follows: each interval $I(n_1)$ with n_1 odd is divided into intervals $I(n_1, n_2)$ so that the length of intervals with n_2 even is d_2, and the length of intervals with n_2 odd is r_2, and $d_2=(2^2+1)r_2=5r_2$. Each odd indexed interval $I(n_1)$ is divided into $2^{3.2} + 2^{2.2+1} + 2^{2+2} + 7=119$ subintervals of equal length r_2. Let $c_2=1/119$ and note that $r_2=c_1c_2$. In general, stage k intervals are denoted by $I(n_1,\ldots,n_k)$, where $0\leqslant n_i<N_i$, $N_i=2.2^{2i}+7$, $i=1,2,\ldots,k$, and for $i<k$ all n_i are odd. The intervals $I(n_1,\ldots, n_k)$ divide the interval $I(n_1,\ldots,n_{k-1})$, where $n_1,\ldots,n_{k-1}$ are fixed and all odd, into $N_k=2.2^{2k}+7$ intervals so that the length of the intervals with n_k even is d_k, and the length of the intervals with n_k odd is r_k, and $d_k=(2^{2k}+1)r_k$. In effect, each interval with odd n_i, $i<k$, is divided into $2^{3k} + 2^{2k+1} + 2^{k+2}+7$ equal length intervals of length r_k. Each $I(n_1,\ldots,n_k)$ with n_k even contains 2^k+1 of these; each $I(n_1,\ldots,n_k)$ with n_k odd consists of one of them. Let $c_k=1/(2^{3k} + 2^{2k+1} + 2^{k+2} + 7)$; then $r_k=c_1,c_2,\ldots c_k$. We also determine the positions of the left endpoints of the intervals $I(n_1,\ldots,n_k)$. If n_k is even, the left endpoint of $I(n_1,\ldots,n_k)$ is $(2^{k-1}+1)n_kr_k$ plus the value of the left endpoint of the interval $I(n_1,\ldots,n_{k-1})$. If n_k is odd, the left endpoint of $I(n_1,\ldots,n_k)$ is $(2^{k-1}+1)n_kr_k+2^{k-1}r_k$ plus the value of the left endpoint of $I(n_1,\ldots,n_{k-1})$.

Coefficients associated with subintervals:

Since in our construction the heights of the jagged functions depend on the positions of the subintervals above which they are placed, we define the coefficients P_{n_k} for stage k as follows: If n_k is even,

$$P_{n_k}=(n_k+1)/2^{2k} \text{ for } 0\leqslant n_k\leqslant 2^{2k}-2$$
$$P_{n_k}=(2^{2k+1}+7-n_k)/2^{2k} \text{ for } 2^{2k}+8\leqslant n_k\leqslant 2.2^{2k}+6$$
$$P_{n_k}=1 \text{ for } n_k=2^{2k},\ 2^{2k}+2,\ 2^{2k}+4,\ 2^{2k}+6.$$

If n_k is odd, $I(n_1,\ldots,n_k)$ is adjacent to two intervals $I(n_1,\ldots,\ n_{k-1},\ n_k-1)$ and $I(n_1,\ldots,n_{k-1},n_k+1)$ with $n_k\pm 1$ even. Then define

$$P_{n_k}=(P_{n_k-1}+P_{n_k+1})/2.$$

<u>Jagged functions for different stages</u>:

Recall the function H(x) linear between any two consecutive of the following points (0,0), (1/4, 1), (3/4,-1), and (1,0), and H(x)=0 for x outside [0,1]. On the other hand, because of the successive partitions performed above, each x in [0,1] has an expansion

$$x=\sum_{k=1}^{\infty} a_k r_k,\ a_k=0,1,\ldots,2^{3k}+2^{2k+1}+2^{k+2}+6.$$

Since each interval $I(n_1,\ldots,n_k)$ with n_k even contains 2^k+1 intervals of length r_k, a_k can have 2^k+1 differenct values in the interval $I(n_1,\ldots,n_k)$. Note that those values of a_k may be even or odd. So in the expansion $a_k r_k$ of x, a_k can have values between 0 and $2^{3k}+2^{2k+1}+2^{k+2}+6$. If $a_k=(2^{k-1}+1)\ n_k\ r_k+2^{k-1}r_k$ where n_k is odd, then a_k is the left endpoint of $I(n_1,\ldots,n_k)$. Otherwise a_k belongs to an interval $I(n_1,\ldots,n_k)$ with n_k even.

In the partition of stage 1, define $f_1(x)=H[(x-e_{n_1})/d_1]P_{n_1}$ if a_1 belongs to $I(n_1)$ with n_1 even, and e_{n_1} is the left endpoint of $I(n_1)$, $f_1(x)=0$ if a_1 is the left endpoint of an odd indexed interval.

In the partition of stage k, define

$f_k(x)=H[(x-e_{n_1\ldots n_k})/d_k]P_{n_1}P_{n_2}\ldots P_{n_k}$ if $a_1,\ldots,a_{k-1}$ are the left endpoints of odd indexed intervals with $n_1=a_1,\ldots,n_{k-1}=a_{k-1}$, and $e_{n_1\ldots n_k}$ is the left endpoint of $I(n_1,\ldots,n_k)$ with n_k even which x belongs to, $f_k(x)=0$ if all $a_1,\ldots,a_k$ are the left endpoints of odd indexed intervals.

The function f(x) is then defined as follows:

$$f(x)=\sum_{k=1}^{\infty}f_k(x).$$

Note that each function $f_k(x)$ is continuous on [0,1], and f(x) is measurable and in Baire class one since f(x) is the limit of the sequence of continuous functions

$$h_n(x)=\sum_{k=1}^{n}f_k(x).$$

<u>Points of continuity and points of discontinuity of f(x)</u>:

For each x in [0,1], consider the expansion

$$x=\sum_{k=1}^{\infty} a_k r_k,\quad a_k=0,1,\ldots,2^{3k}+2^{2k+1}+2^{k+2}+6.$$

If there are a_p's belonging to an interval $I(n_1,\ldots,n_p)$ with n_p even, n_i odd for i p, then x is a point of continuity for f(x) since $f(x)=f_p(x)$ in this interval and $f_p(x)$ is a continuous function.

Suppose all a_k's are left endpoints of odd indexed intervals. Then we associate with x the infinite product

$$P(x)=\prod_{k=1}^{\infty} P_k(x).$$

For simplicity, we write

$$P_k(x)=P_{n_k}(x).$$

Thus
$$P(x)=\prod_{k=1}^{\infty} P_k(x).$$

If $P(x)=0$, then x is a point of continuity for $f(x)$ because the oscillation of f at x is 0.

If $P(x)>0$, then x is a point of discontinuity for $f(x)$ because the oscillation of f at x is $2P(x)$.

The set of points of discontinuity of f in $[0,1]$ is uncountable since it contains the set $E=\{x:P_k(x)=1,\ k=1,2,\dots\}$, which is a perfect set, and at every x in E, the oscillation of f is 2.

<u>Symmetry of f(x)</u>:

It is clear that f is symmetric at any of its points of continuity [7].

Let x be a point of discontinuity for f. Then $f(x)=0$ and $P(x)>0$. It is well-known [3, p. 96] that if the infinite product $\prod_{k=1}^{\infty} P_k(x)$ is positive, then $\sum_{k=1}^{\infty} 1-P_k(x)<\infty$. Given $\varepsilon>0$, choose N so that $\sum_{k=N}^{\infty} 2(1-P_k(x))+8.2^{-2N-1}<\varepsilon.P(x)$

where the coefficient 8 in the term 8.2^{-2N-1} comes from the fact that the seven middle intervals in the partition of each $I(n_1,\ldots,n_{k-1})$ with $n_1,\ldots,n_{k-1}$ odd have, by definition, the same coefficient. Fix $\delta > 0$ so that, for $|h| < \delta$, x+h and x-h both belong to the interval $I(n_1,\ldots,n_N)$. We will say that f at t is determined at the mth stage if t is in $I(n_1', n_2',\ldots, n_m')$ with n_m' even and n_i' odd for $i < m$. There are three possibilities for a given pair x-h and x+h.

a) The function f at x-h and x+h is never determined. In this case

$$f(x+h)=f(x-h)=0.$$

So $$f(x+h)+f(x-h)-2f(x)=0$$

b) The function f is determined at the mth stage at one point but not determined at that stage or an earlier stage at the other point. Without loss of generality, suppose f is determined at x+h at the mth stage but not determined at x-h at the mth or an earlier stage. Then $m > N$, and, by the definition of N:

$$P_m(x) < 1-\varepsilon/2 \text{ and } P_{m+1}(x) < 1-\varepsilon/2.$$

Let a be the center of $I(n_1',\ldots,n_{m-1}')$, to which x-h belongs, and suppose x-h belongs to $I(n_1',\ldots,n_m')$, n_i' odd, $i=1,\ldots,m$. Let b be the center of $I(n_1,\ldots,n_{m-1})$, to which x belongs, and suppose x belongs to $I(n_1,\ldots,n_m)$, n_i odd, $i=1,\ldots,m$. Let c be the center of $I(n_1'',\ldots,n_{m-1}'')$, to which x+h belongs, and suppose x+h belongs to $I(n_1'',\ldots,n_m'')$, n_m'' even, n_i'' odd for $i<m$. Then $c-b=b-a$, and the

common length of the intervals $I(n_1',\ldots,n_{m-1}')$, $I(n_1,\ldots,n_{m-1})$ and $I(n_1'',\ldots,n_{m-1}'')$ is r_{m-1}. Since $P_m(x) < 1-\varepsilon/2$, we have $|x-b| < \varepsilon r_{m-1}$. Let u be the center of $I(n_1,\ldots,n_m)$. Then $|x-u| < \varepsilon r_m$. Let v be the center of $I(n_1',\ldots,n_m')$. Then it is <u>impossible</u> that

$$|x-h-v| < (1/2-2\varepsilon)r_m$$

because then x+h would lie in $I(n_1'',\ldots,n_{m-1}'',k)$ with k odd, whose center is 2b-v and f(x+h) would not be determined at the mth stage. That is, x-h is very close to the endpoints of $I(n_1',\ldots,n_m')$, n_i odd, $i=1,\ldots,m$, and x+h is very close to the endpoints of $I(n_1'',\ldots,n_m'')$, n_m'' even, n_i'' odd for $i<m$. Moreover, the distance from x+h to $I(n_1'',\ldots,n_{m-1}'',n_m'')$ is less than $3\varepsilon r_m$ and thus

$$\begin{aligned} f(x+h) &\leqslant H\ (3\varepsilon r_m/d_m) \\ &\leqslant 4.3\varepsilon r_m/d_m < 12\varepsilon/2^m+1 \end{aligned}$$

because the slope of H(x) is 4 for $x<1/4$ or $x>3/4$. On the other hand, the minimum distance from x-h to the intervals $I(n_1',\ldots,n_{m\pm 1}')$ adjacent to $I(n_1',\ldots,n_m')$ is less than $2\varepsilon r_m$. So by the definition of the coefficents P_k and by proportionality,

$$P_{m+1}(x-h) < 2\varepsilon/(2^{3m-1}+2^{2m})$$

because there are $2^{3m-1}+2^{2m}$ intervals of length r_m on each side of the seven middle intervals of a copy at the mth stage. Thus

$$|f(x-h)| < 2\varepsilon/(2^{3m-1}+2^{2m}),$$

and $|f(x+h)+f(x-h)-2f(x)| < 12\ \varepsilon\ /(2^m+1)+2\ \varepsilon\ /(2^{3m-1}+2^{2m})$.

c) The function f is determined at x+h and x-h at the same stage; say the mth stage. Let

$$Q_M(t)=\prod_{k=N}^{M} P_k(t).$$

Then for $M=N, N+1,\dots,m-1$,

$$(*)\quad \left|Q_M(x+h)-Q_M(x-h)\right| \leqslant \sum_{k=N}^{M} 2(1-P_k(x))+8.2^{-2k-1}$$

will be proved by induction on M.

First we prove (*) is true for $M=N$. Suppose $x+h$ and $x-h$ belong to the same copy at stage N in the interval $I(n_1,\dots,n_{N-1})$ with $n_1,\dots,n_{N-1}$ odd.

If $x+h$ and $x-h$ belong to two intervals $I(n_1,\dots,n_{N-1},n_N)$ and $I(n_1,\dots,n_{N-1},n_N+2)$ with two consecutive even indices n_N and n_N+2, then by the definition of the coefficients P_{n_k}, we have

$$\left|P_N(x+h)-P_N(x-h)\right| = \left|P_{n_N}-P_{n_N+2}\right| < 2/2^{2N}=2^{-2N-1}.$$

Let $\quad a=2^{3N-1}+2^{2N}$ and $b=2^{3N-1}+2^{2N}+2^{N+2}+7$

and let e be the left endpoint of $I(n_1,\dots,N_{N-1})$.

If $e<x-h<x<x+h<e+ar_N$, then

$$1-P_N(x) \geqslant P_N(x+h)-P_N(x)$$

$$1-P_N(x) \geqslant P_N(x)-P_N(x-h)$$

Thus $$2(1-P_N(x)) \geqslant P_N(x+h)-P_N(x-h).$$

If $e<x-h<x\leqslant e+ar_N<x+h \leqslant e+br_N$, then

$$2(1-P_N(x)) \geqslant P_N(x+h)-P_N(x-h)+8.2^{-2N-1}.$$

If $e < x-h<x \leqslant e+br_N < x+h$, then with $k=e+br_N-x$,

$$\left|P_N(x+h)-P_N(x-h)\right| = \left|P_N(x+k)-P_N(x-k)\right| \leqslant 2(1-P_N(x))+8.2^{-2N-1}$$

So in the case in which $x+h$ and $x-h$ belong to the same copy at stage N, we have

$$\left|Q_N(x+h)-Q_N(x-h)\right| \leqslant 2(1-P_N(x))+8.2^{-2N-1}$$

Now suppose x+h and x-h belong to two differnet copies of the Nth stage. Let $I(n_1',\ldots,n_N')$ with $n_1',\ldots,n_N'$ odd, be the interval to which x-h belongs; $I(n_1,\ldots,n_N)$ with $n_1,\ldots,n_N$ odd, the interval to which x belongs; and $I(n_1'',\ldots,n_N'')$ with $n_1'',\ldots,n_N''$ odd, the interval to which x+h belongs. Since the partitions of the intervals $I(n_1',\ldots,n_{N-1}')$, $I(n_1,\ldots,n_{N-1})$ and $I(n_1'',\ldots,n_{N-1}'')$ are similar, consider the subintervals $I(n_1,\ldots,n_{N-1},n_N')$ and $I(n_1,\ldots,n_{N-1},n_N'')$, which correspond respectively to $I(n_1',\ldots,n_N')$ and $I(n_1'',\ldots,n_N'')$.

Then $$P_N(x-h)=P_{n_N'} \text{ and } P_N(x+h)=P_{n_N''}$$

where $P_{n_N'}$ and $P_{n_N''}$ are the coefficients associated with the intervals $I(n_1,\ldots n_{N-1},n_N')$ and $I(n_1,\ldots,n_{N-1},n_N'')$. Thus

$$\begin{aligned}\left|P_N(x+h)-P_N(x-h)\right| &= \left|P_{n_N'}-P_{n_N''}\right| \\ &\leqslant 2(1-P_N(x))+8.2^{-2N-1}\end{aligned}$$

because the intervals $I(n_1,\ldots,n_{N-1},n_N')$ and $I(n_1,\ldots,n_{N-1},n_N'')$ belong to the same copy of stage N. Thus (*) is true for M=N.

Suppose (*) is true for M=p:

$$\left|Q_p(x+h)-Q_p(x-h)\right| \leqslant \sum_{k=N}^{p} 2(1-P_k(x))+8.2^{-2k-1}$$

Then

$$\left|Q_{p+1}(x+h)-Q_{p+1}(x-h)\right| = \left|Q_p(x+h).P_{p+1}(x+h)-Q_p(x-h).P_{p+1}(x-h)\right|$$

$$\leqslant \left|Q_p(x+h)-Q_p(x-h)\right|.P_{p+1}(x+h)+Q_p(x-h).\left|P_{p+1}(x+h)-P_{p+1}(x-h)\right|$$

$$\leqslant \sum_{k=N}^{p} \{2(1-P_k(x))+8.2^{-2k-1}\}+2(1-P_{p+1}(x))+2^{-2(p+1)-1}$$

because of the previous results.

Now, since f is determined at x-h and x+h at the mth stage, let D equal the distance from $I(n_1,\ldots,n_m)$ to $I(n_1'',\ldots,n_m'')$ and note that D also equals the distance from $I(n_1,\ldots,n_m)$ to $I(n_1',\ldots,n_m')$. Let $k=h-D-d_m$. Then

$$Q_m(x).f_m(x+h)=Q_m(x+h).f_m(x+k)$$

and $Q_m(x).f_m(x-h)=Q_m(x-h).f_m(x-k)$.

Finally,

$$\left|f_m(x+h)+f_m(x-h)\right| \leqslant \frac{1}{Q_m(x)}\left|Qm\ (x+h).f_m(x+k)+Q_m(x-h).f_m(x-k)\right|$$

$$\leqslant \frac{1}{1-\varepsilon}\ \left|Q_m(x+h)-Q_m(x-h)\right|.\ f_m(x+k)\ \ +Q_m(x-h).\left|f_m(x+k)+f_m(x-k)\right|$$

$$\leqslant \frac{1}{1-\varepsilon}(\varepsilon+2^{-m-2})$$

since $\left|f_m(x+k)+f_m(x-k)\right| \leqslant 4\ r_m/d_m=4/(2^m+1)<2^{-m-2}<2^{-(N+2)}$.

Thus for N sufficiently large, $\left|f(x+h)+f(x-h)-2f(x)\right|$ is small and $f(x+h)+f(x+h)-2f(x)=o(1)$.

<u>Notes</u>: 1) Preiss' example of a function satisfying

$$\lim_{h\to 0} f(x+h)-f(x-h)=0$$

depends on the existence of a certain type of convergent Fourier series [6].

An explicit example can be obtained by the function

$$g(x)=\left|f(x)\right|$$

where $f(x)$ is the function just studied. For at each point of continuity of $f(x)$, $g(x)$ is also continuous. At points of discontinuity of $f(x)$, $f(x)=0$. Thus at points of discontinuity, we have

$$|g(x+h)-g(x-h)|=\big|\,|f(x+h)|-|f(x-h)|\,\big|$$

$$=|f(x+h)+f(x-h)| \text{ if } f(x+h) \text{ and } f(x-h) \text{ have opposite}$$

signs

$$=|f(x+h)-f(x-h)|\leqslant|f(x+h)+f(x-h)| \text{ if } f(x+h) \text{ and}$$

$f(x-h)$ have the same sign.

Thus $\lim\limits_{h\to 0} g(x+h)-g(x-h)=0.$

2) Since the uniform limit of a sequence of symmetric functions is symmetric [7], it is possible to construct such a function which has c points of discontinuity in every interval. In fact, let $g_1(x)$ be the function $f(x)$, which was constructed. Assume $f(x)=0$ if x is not in $[0,1]$. Let $I_1=[a_1, b_1]$ be one of the largest intervals in $[0,1]$, on which g_1 has no points of discontinuity. Let

$$g_2(x)=g_1(x)+2^{-4}f[(x-a_1)/(b_1-a_1)].$$

In general, let $I_{n-1}=[a_{n-1},b_{n-1}]$ be one of the largest intervals in $[0,1]$, on which g_{n-1} has no points of discontinuity, and let

$$g_n(x)=g_{n-1}(x)+2^{-2^n}f[(x-a_{n-1})/(b_{n-1}-a_{n-1})].$$

Then $g_n(x)$ converges uniformly to a function $f(x)$. Since each successive function $g_n(x)$ has c points of discontinuity with oscillation of 2^{-2^n} on $[a_n, b_n]$, and since

$$2^{-2^n} > \sum_{i=n+1}^{\infty} 2^{-2^i},$$

these are also points of discontinuity of g. Thus, g has c points of discontinuity in each interval.

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