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The Powers and their Bernstein Polynomials

We wish to establish a relationship between the powers and Their Bernstein polynomials.

We start with a brief review of Stirling numbers because of their importance in developing the theory.

Denote Stirling numbers of the first kind by $s(m,r)$.

$$\begin{cases} s(1,1) = 1 \quad \text{while} \quad s(1,r) = 0 \quad \text{for} \quad r \neq 1, \\ s(m,r-1) - m\,s(m,r) = s(m+1,r). \end{cases} \tag{1}$$

Denote Stirling numbers of the second kind by $S(m,r)$.

$$\begin{cases} S(1,1) = 1 \quad \text{while} \quad S(1,r) = 0 \quad \text{for} \quad r \neq 1, \\ S(m,r-1) + r\,S(m,r) = S(m+1,r). \end{cases} \tag{2}$$

We write them in matrix form: $A = (s(m,r))$, $S = (S(m,r))$

$$A = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & \bigcirc & \\ 2 & -3 & 1 & & \\ -6 & 11 & -6 & 1 & \\ 24 & -50 & 35 & -10 & 1 \\ \hdashline \end{pmatrix} \qquad S = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & \bigcirc & \\ 1 & 3 & 1 & & \\ 1 & 7 & 6 & 1 & \\ 1 & 15 & 25 & 10 & 1 \\ \hdashline \end{pmatrix} \tag{3}$$

They are inverse to each other.

$$A^{-1} = S \, , \quad S^{-1} = A \, , \quad AS = SA = I \tag{4}$$

Some mathematicians define Stirling numbers of the second kind to be

$$S(m,r) = \frac{1}{r!}\sum_{k=1}^{r}(-1)^k\binom{r}{k}(r-k)^m = \frac{(-1)^r}{r!}\sum_{k=1}^{r}(-1)^k\binom{r}{k}\,k^m \tag{5}$$

The Bernstein polynomials are important in approximation theory, Fourier theory, differential equation theory, etc.

For $f = [0,1] \to R$, define the Bernstein polynomial of $f(x)$ to be

$$B_n(x,f(x)) = \sum_{k=0}^{n} f(\tfrac{k}{n})\binom{n}{k} x^k(1-x)^{n-k} \tag{6}$$

A common way to compute $B_n(x,f(x))$ for powers $f(x) = 1,x,x^2$ is as follows.

$$(u+v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k} .$$

Let $u = x$, $v = 1-x$. We get $1 = \sum_{k=0}^{n} (1)\binom{n}{k} x^k(1-x)^{n-k}$.

By definition, $B_n(x,1) = 1$.

Using $1 = B_{n-1}(x,1) = \sum_{j=0}^{n-1} \binom{n-1}{j} x^j(1-x)^{n-1-j}$ and $\binom{n-1}{j} = (\frac{j+1}{n})\binom{n}{j+1}$,

we get

$$x = \sum_{j=0}^{n-1} (\tfrac{j+1}{n})\binom{n}{j+1} x^{j+1}(1-x)^{n-(j+1)} = \sum_{k=0}^{n} (\tfrac{k}{n})\binom{n}{k} x^k(1-x)^{n-k}.$$

By definition, $B_n(x,x) = x$.

Similarly, from $x = B_{n-1}(x,x) = \sum_{j=0}^{n-1} (\frac{j}{n-1})\binom{n-1}{j} x^j(1-x)^{n-1-j} =$

$= \sum_{j=1}^{n-1} \binom{n-2}{j-1} x^j(1-x)^{n-(j+1)}$, we get

$$x^2 = \sum_{j=1}^{n-1} \binom{n-2}{j-1} x^{j+1}(1-x)^{n-(j+1)} = \sum_{k=2}^{n} \binom{n-2}{k-2} x^k(1-x)^{n-k}.$$

Since $(1-\frac{1}{n})\binom{n-2}{k-2} = \frac{(n-1)}{n} \cdot \frac{k(k-1)}{n(n-1)}\binom{n}{k} = [(\frac{k}{n})^2 - \frac{1}{n}(\frac{k}{n})]\binom{n}{k}$,

$$(1-\frac{1}{n})\ x^2 = \sum_{k=0}^{n} (\frac{k}{n})^2 \binom{n}{k} x^k(1-x)^{n-k} - \frac{1}{n}\sum_{k=0}^{n} (\frac{k}{n})\binom{n}{k} x^k(1-x)^{n-k}$$

$$= B_n(x,x^2) - \frac{1}{n} B_n(x,x).$$

Thus $B_n(x,x^2) = \frac{x}{n} + (1-\frac{1}{n})\ x^2.$

We could proceed in a similar manner to obtain expressions for higher degree, but this is a cumbersome procedure that may not lead us to a general expression for any term. To solve the general problem, we obtain the following result.

<u>Theorem</u>: Suppose $B^{(m)} = B_n(x,x^m)$ is the Bernstein polynomial of x^m.

Then $$B^{(m)}(x) = \sum_{r=1}^{m} S(m,r)\ \frac{n(n-1)\cdots(n-r+1)}{n^m}\ x^r,\quad m \geq 1\ ,\ n \geq 1, \tag{7}$$

$$x^m = \sum_{r=1}^{m} \frac{n^r}{n(n-1)\cdots(n-m+1)}\ s(m,r)\ B^{(r)}\quad ,\quad n \geq m \geq 1, \tag{8}$$

The sets $\{1,x,x^2,x^3,\cdots\}$ and $\{B^{(0)},B^{(1)},B^{(2)},B^{(3)},\cdots\}$ are both bases of the same vector space. The change of basis is

$$\mathcal{B} = \mathcal{N}^{-1}\ \mathcal{S}\mathcal{M}\mathcal{X}\ ,\quad m \geq 1\ ,\ n \geq 1, \tag{9}$$

$$\mathcal{X} = \mathcal{M}^{-1}\ \mathcal{A}\mathcal{N}\mathcal{B}\ ,\quad n \geq m \geq 1, \tag{10}$$

where $\mathcal{A}$ and $\mathcal{S}$ are shown as (3) and (4), and

$$\mathcal{B} = \begin{pmatrix} B^{(1)} \\ \vdots \\ B^{(m)} \\ \vdots \end{pmatrix}\ ,\quad \mathcal{X} = \begin{pmatrix} x \\ \vdots \\ x^m \\ \vdots \end{pmatrix}\ ,\quad \mathcal{M} = \begin{pmatrix} 1 & & & \bigcirc \\ & n-1 & & \\ & & (n-1)(n-2) & \\ \bigcirc & & & \ddots \end{pmatrix}\ ,$$

$$\mathcal{M}^{-1} = \begin{pmatrix} 1 & & & \bigcirc \\ & \frac{1}{n-1} & & \\ & & \frac{1}{(n-1)(n-2)} & \\ \bigcirc & & & \ddots \end{pmatrix},\ m \geq n,\quad \mathcal{N} = \begin{pmatrix} 1 & & \bigcirc \\ & n & \\ & & n^2 \\ \bigcirc & & & \ddots \end{pmatrix},\quad \mathcal{N}^{-1} = \begin{pmatrix} 1 & & \bigcirc \\ & \frac{1}{n} & \\ & & \frac{1}{n^2} \\ \bigcirc & & & \ddots \end{pmatrix}$$

Proof: We first show (7) which is essential.

$$B^{(m)} = \sum_{k=0}^{n} (\tfrac{k}{n})^m \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=1}^{n} (\tfrac{k}{n})^m \binom{n}{k} x^k \sum_{t=0}^{n-k} (-1)^t \binom{n-k}{t} x^t$$

$$B^{(m)} = \sum_{k=1}^{n} \sum_{t=0}^{n-k} (\tfrac{k}{n})^m \binom{n}{k+t} (-1)^t \binom{k+t}{k} x^{k+t}, \quad \text{since} \quad \binom{n}{k}\binom{n-k}{t} = \binom{n}{k+t}\binom{k+t}{k}.$$

Let $r = k+t$. Then

$$B^{(m)} = \sum_{r=1}^{n} \sum_{k=1}^{r} \frac{k^m}{n^m} \binom{n}{r} (-1)^{r+k} \binom{r}{k} x^r = \sum_{r=1}^{n} \frac{r!}{n^m} \binom{n}{r} \left[\frac{(-1)^r}{r!} \sum_{k=1}^{r} (-1)^k \binom{r}{k} k^m\right] x^r.$$

We thus obtain (7) by (5).

Now consider $\{1, x, x^2, x^3, \cdots\}$ as the basis of the space. Omit the simple case $B^{(0)} = 1$. Writing the result for $B^{(m)}$ in matrix form and using decomposition

$$\begin{pmatrix} B^{(1)} \\ B^{(2)} \\ B^{(3)} \\ \vdots \end{pmatrix} = \begin{pmatrix} S(1,1) & 0 & 0 & \cdots \\ S(2,1)\frac{1}{n}, & S(2,2)\frac{n-1}{n}, & 0 & \cdots \\ S(3,1)\frac{1}{n^2}, & S(3,2)\frac{n-1}{n^2}, & S(3,3)\frac{(n-1)(n-2)}{n^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & & & \\ & \frac{1}{n} & & \\ & & \frac{1}{n^2} & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} S(1,1) & 0 & 0 & \cdots \\ S(2,1), & S(2,2) & 0 & \cdots \\ S(3,1), & S(3,2), & S(3,3) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} 1 & & & \\ & n-1 & & \\ & & (n-1)(n-2) & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix}$$

or $\quad B = N^{-1} S M X. \quad$ (11)

If $n \geq m \geq 1$, then M is nonsingular.

So $\quad X = (N^{-1} S M)^{-1} B = M^{-1} A N B. \quad$ (12)

And we obtain expression (8) for the components of the vector X.

Thus our theorem is proved.

For $m \geq 2$, we rewrite (7) and (8) as

$$B^{(m)} = \frac{x}{n^{m-1}} + \sum_{r=2}^{m} S(m,r)(1-\frac{1}{n})\cdots(1-\frac{r-1}{n}) \frac{x^r}{n^{m-r}} , \tag{13}$$

$$x^m = \frac{1}{(n-1)\cdots(n-m+1)} \sum_{r=1}^{m} S(m,r)\, n^{r-1}\, B^{(r)} . \tag{14}$$

$$B^{(1)} = x$$

$$B^{(2)} = \frac{x}{n} + (1-\frac{1}{n})x^2$$

$$B^{(3)} = \frac{x}{n^2} + 3(1-\frac{1}{n})\frac{x^2}{n} + (1-\frac{1}{n})(1-\frac{2}{n})x^3$$

$$B^{(4)} = \frac{x}{n^3} + 7(1-\frac{1}{n})\frac{x^2}{n^2} + 6(1-\frac{1}{n})(1-\frac{2}{n})\frac{x^3}{n} + (1-\frac{1}{n}) \cdots (1-\frac{3}{n})x^4$$

$$B^{(5)} = \frac{x}{n^4} + 15(1-\frac{1}{n})\frac{x^2}{n^3} + 25(1-\frac{1}{n})(1-\frac{2}{n})\frac{x^3}{n^2} + 10(1-\frac{1}{n})\cdots(1-\frac{3}{n})\frac{x^4}{n} + (1-\frac{1}{n})\cdots(1-\frac{4}{n})x^5$$

$$x = B^{(1)}$$

$$x^2 = \frac{1}{n-1} (-B^{(1)} + n\,B^{(2)})$$

$$x^3 = \frac{1}{(n-1)(n-2)} (2B^{(1)} - 3\,n\,B^{(2)} + n^2\,B^{(3)})$$

$$x^4 = \frac{1}{(n-1)\cdots(n-3)} (-6B^{(1)} + 11\,n\,B^{(2)} - 6\,n^2\,B^{(3)} + n^3\,B^{(4)})$$

$$x^5 = \frac{1}{(n-1)\cdots(n-4)} (24B^{(1)} - 50\,n\,B^{(2)} + 35\,n^2\,B^{(3)} - 10\,n^3\,B^{(4)} + n^4\,B^{(5)})$$

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