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On Foran's Property (M) and its relation to Lusin's Property (N)

In [1], J.Foran has constructed a continuous function F on [0,1] satisfying on [0,1] the Foran property (M), but which does not satisfy Lusin's property (N). In Theorem 3 we shall show that such a function may be obtained by using a result due by S.Mazurkiewicz more than fifty years ago [2]. Our strategy will be the following. We shall construct a continuous function F having on [0,1] Lusin's property (N) such that for any linear nonconstant function g, the function F+g has the Foran property (M), but does not have Lusin's property (N). Our idea is inspired by the S.Mazurkiewicz result [2] asserting the existence of a continuous function F having on [0,1] Lusin's property (N) such that F+g has the property (N) for no linear nonconstant function g.

As will become apparent, S.Mazurkiewicz' paper [2] anticipates implicitly the Foran property (M).

In Theorem 2 of our paper we indicate a new, shorter way to obtain the above-mentioned Mazurkiewicz Theorem.

Definition. A continuous function F is said to satisfy property (M) providing F is AC on any set E on which it is VB ([1]).

Remark 1. Lusin's condition (N) implies property (M), but the converse is not true. This was shown by Foran in [1] . In the present paper we give an example which is another proof of Foran's result.

Theorem 1. If F is a continuous function satisfying ACG on [a,b], and if G is a continuous function satisfying (M) on [a,b], then every linear combination of F and G, also satisfies (M) on [a,b].

Proof. Let $H(x) = F(x) + G(x)$, $x \in [a,b]$. Suppose that $H(x)$ is VB on a set $E \subset [a,b]$ (E may be supposed closed). Since F is ACG on [a,b], there is a sequence of sets E_n such that $E = \bigcup E_n$ and F is AC on E_n . Now it is clear that $G(x)$ is VB on each E_n . Since G satisfies (M) on [a,b], it follows that G is ACG on E. Hence H is ACG on E. By the Banach-Zarecki Theorem ([3],pp.227), H is AC on E.

We shall now give a new, simplified proof of the above-mentioned S.Mazurkiewicz Theorem.

Theorem 2. (S.Mazurkiewicz). There exists a continuous function F having Lusin's property (N) on [0,1] such that F+g has property (N) for no linear nonconstant function g .

Proof. We shall first construct a function F and we shall show that it has the properties required by Theorem 2. Let $n_1 \geqslant 2$ and let

$$n_k = 2^{k-1}(4n_1+5)\dots(4n_{k-1}+5) \quad , \ k \geqslant 2$$

$$m_k = 2(n_k+1)(2n_k+1) + n_k \quad , \ k \geqslant 1$$

$$a_k = 2/((2m_1+1)\dots(2m_k+1)) \quad , \ k \geqslant 1$$

$$b_k = 2/((2n_1+1)\dots(2n_k+1)) \quad , \ k \geqslant 1 \ .$$

Let $P = \{x : x = \sum a_i p_i \ , \ p_i = 0,\dots,m_i\}$ and

$Q = \{y : y = \sum b_i q_i \ , \ q_i = 0,\dots,n_i\}$.

Clearly $|P| = |Q| = 0$. We denote by $r(x;y)$ the remainder of the quotient x/y. Let F be a function defined on $[0,1]$ as follows. For each $x \in P$, $F(x) = F(\sum a_i p_i(x)) = \sum b_i r(p_i(x);n_i+1)$. Then F is continuous on P and, by extending F linearly on each interval contiguous to P, one has F defined and continuous on $[0,1]$. Clearly $F(P) = Q$. Hence $|F(P)| = |Q| = 0$ and F fulfils (N). For any real $t \neq 0$ let $G_t(x) = F(x) + tx$, $x \in [0,1]$. Let

$I_{i_1,\dots,i_k} = [c_{i_1,\dots,i_k} \ , \ d_{i_1,\dots,i_k}]$, where

$$c_{i_1,\dots,i_k} = \sum_{p=1}^{k} i_p a_p \quad \text{and} \quad d_{i_1,\dots,i_k} = c_{i_1,\dots,i_k} + \sum_{p \geqslant n+1} m_p a_p \quad , \ i_j = 0,\dots,m_j \ ; \ j = 1,\dots,k \ .$$

Let $t > 0$. Then there is a natural number k such that $t \in [1/2^k \ , \ 2^k]$. Since $2^k(2n_k+1) < b_k/a_k < (1/2^k)(m_k-n_k-1)$, we have

$G_t(I_{i_1,\dots,i_{k-1},(n_k+1)j+i}) \cap$

$\cap G_t(I_{i_1,\dots,i_{k-1},(n_k+1)(j+1)+i}) \neq \emptyset$, $i=0,\dots,n_k$, $j = 0,\dots,4n_k+1$ and

$G_t(c_{i_1,\dots,i_{k-1},i+1}) < G_t(c_{i_1,\dots,i_{k-1},m_k-n_k+i})$,

$i = 0,\dots,n_k-1$. Now $G_t(I_{i_1,\dots,i_{k-1}} \cap P) =$

$= G_t(I_{i_1,\dots,i_{k-1}})$. Since $|G_t(I_{i_1,\dots,i_{k-1}})| > 0$,

G_t does not satisfy (N) .

Let $t<0$. Then there is a natural number k, such that $-t \in [1/2^k, 2^k]$. Since $2^k(2n_k+3) < b_k/a_k <$

$< (1/2^k)(m_k-n_k+1)$, we have

$G_t(I_{i_1,\dots,i_{k-1},(n_k+1)j+i}) \cap$

$\cap G_t(I_{i_1,\dots,i_{k-1},(n_k+1)(j+1)+i}) \neq \emptyset$,

$i = 0,\dots,n_k$, $j = 0,\dots,4n_k+1$, and

$G_t(c_{i_1,\dots,i_{k-1},i}) > G_t(c_{i_1,\dots,i_{k-1},m_k-n_k+i+1})$,

$i = 0,\dots,n_k-1$. Now $G_t(I_{i_1,\dots,i_{k-1}} \cap P) =$

$= G_t(I_{i_1,\dots,i_{k-1}})$. Hence G_t does not satisfy (N).

The aim of the next theorem is to get another proof for the existence of a function considered in the Foran example.

<u>Theorem 3. The function G_t considered in the proof of Theorem 2 has the Foran property</u> (M), <u>but does not have Lusin's property</u> (N).

Proof. Since condition (N) implies the Foran property (M), by Theorem 1, it follows that G_t fulfils (M). Clearly, by Theorem 2, G_t does not satisfy condition (N) .

Remark 2. As can be seen from the proofs of Theorems 2 and 3, we get an uncountable class of functions of the type envisaged by Foran.

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References

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