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Porosity Characteristics of Intersection Sets with the Typical Continuous Function

In 1931, Banach and Mazurkiewicz proved independently that the 'typical continuous function' is nowhere differentiable. (We use the term 'typical continuous function' to mean that the set of functions which exhibit the property we are discussing is residual in $\mathcal{C}[0,1]$.) In 1933 and 1934 Jarnik proved that the typical continuous function has every extended real number as a derived number, everywhere; and every extended real number as an essential derived number, almost everywhere.

We will come back to these results to illustrate some applications of the present work: in which we ask for what fixed family of functions $\mathfrak{F}$ and 'smallness' P is the set $\{f \in \mathcal{C}[0,1] \mid$ for every $g \in \mathfrak{F}$ $\{t: f(t) = g(t)\}$ is $P\}$ residual? That is, for what $\mathfrak{F}$ and P does the typical continuous function intersect every function in $\mathfrak{F}$ in a P set?

For example, in 1961, Goffman showed that for any modulus of continuity σ and corresponding equicontinuous class $\mathcal{C}(\sigma)$, the typical continuous function intersects every function in $\mathcal{C}(\sigma)$ in a measure zero set. In 1981, Thomson showed that every level set of the typical continuous function is bilaterally strongly porous; (in this case the family of functions is all horizontal lines). In 1982 Bruckner and myself showed that for any σ-compact family $\mathfrak{F}$, the typical continuous function intersects every function in $\mathfrak{F}$ in a bilaterally strongly porous set.

In order to strengthen the 'smallness' P, (much as bilateral strong porosity strengthens measure zero), we will consider the following generalized porosity:

Def: Let ϕ be a strictly increasing continuous function with $\phi(0) = 0$. Say a set $B \subseteq \mathbb{R}$ is ϕ-porous at x if there exists a sequence of intervals $\{I_n\}$ with $I_n \subseteq (B \cup \{x\})^c$ so that $I_n \to x$ and $d(x, I_n) < \phi(|I_n|)$ $\forall n$. Similarly define right, left, and bilateral porosity at x. Say B is ϕ-porous if B is ϕ-porous at all of its points. If $\Phi = \{\phi_\alpha | \alpha \in J\}$ (here, J is an open interval) is a family of porosity functions, say B is strongly Φ-porous if B is ϕ_α-porous $\forall \alpha \in J$. We will call Φ ordered if $\alpha, \beta \in J$ with $\alpha < \beta$ imply there exists a $\delta > 0$ so that $\phi_\alpha(x) < \phi_\beta(x)$ $\forall x \in (0, \delta)$.

Some examples of porosity families:

1. $\Phi = \{\phi_\alpha | \phi_\alpha(x) = \alpha x \quad \alpha \in (0,\infty)\}$. Here, bilateral strong porosity is equivalent to bilateral strong Φ-porosity.
2. $\Phi = \{\phi_\alpha | \phi_\alpha(x) = x^{1/\alpha} \quad \alpha \in (0,1)\}$.

Both the above porosity families are ordered.

If, for an increasing function h, $(h(0) = 0)$, we let μ^h be the (outer) Hausdorff measure associated with h, then it is not hard to show that if a set B is ϕ-porous it is also nowhere dense and $\mu^{\phi^{-1}}(B) = 0$.

One of the results in my thesis is that for any fixed, ordered porosity family Φ, the typical continuous function intersect every Lipshitz function in a bilaterally strongly Φ-porous set. Just recently, Humke and Laczkovich showed that for any fixed, ordered porosity family Φ, the typical continous function intersects every monotonic function in a bilaterally strongly Φ-porous set.

It is not obvious from these examples what the connection is (if any) between the family $\mathfrak{F}$ and the smallness P. The question can be answered in the case where the family $\mathfrak{F}$ is controlled by a modulus of continuity:

Def: For a modulus of continuity σ, let

$$\mathcal{L}(\sigma) = \{g \mid \exists m > 0 \ \forall x,y \ \ |g(x) - g(y)| \leq m\sigma(x - y)\}.$$

Theorem: Let σ be a concave modulus of continuity and Φ be an ordered porostiy family so that $D_+(\sigma\phi^{-1})(0) = 0$. Then, $\{f \in \mathcal{C} \mid \forall g \in \mathcal{L}(\sigma) \ \ \{t: f(t) = g(t)\}$ is strongly Φ-porous$\}$ is residual.

We next give some applications of the above results when the smallness P is taken in the context of Hausdorff dimension: the typical continuous function intersect every function in $\mathcal{L}_\beta$, $(0 < \beta < 1)$, in a set of Hausdorff dimension less than or equal to $1 - \beta$; and if we let Δ be the set of finitely differentiable functions, then the typical continuous function intersect every function in $\mathcal{L} \cup \Delta \cup \mathcal{M}$ (where $\mathcal{M}$ = monotonic functions) in a set of Hausdorff dimension zero. [These results are obtained by choosing the second porosity family in the examples.]

Some more technical applications will lead us back to the original examples of this talk: if E is a system of paths such that there is a porosity function ϕ so that each E_x is (unil.) ϕ-nonporous at x, (i.e. not ϕ-porous at x), then the typical continuous function is nowhere E differentiable; and, the typical continous function has $+$ or $-\infty$ as a left and right essential derived number, everywhere.

We conclude with a result that I worked on for a long time and finally materialized with some help by David Preiss:

Theorem: Let σ be a modulus of continuity and Φ be an ordered and refined* porosity family. If there is a $\phi \in \Phi$ so that

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*Def: Φ is refined if $\{t_n\}$ right ϕ_α-nonporous at zero implies there exists a ϕ_β so $\{t_{2n}\}$ is right ϕ_β-nonporous at zero. Examples 1 and 2 of porosity families are both refined.

$D_+(\sigma\phi^{-1})(0) > 0$, then $\{f \in \mathcal{C} \mid \forall g \in \mathcal{L}(\sigma)$ $\{t: f(t) = g(t)\}$ is strongly Φ-porous$\}$ is empty.