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MULTIPLICATION AND TRANSFORMATION OF DERIVATIVES

We shall investigate finite real functions on the interval $J = [0,1]$. For each system S of functions on J let S^+ [bS] be the system of all nonnegative [bounded] functions in S. Let D [L, C_{ap}] be the system of all derivatives [Lebesgue functions, approximately continuous functions] on J. Let H be the system of all increasing homeomorphisms of J onto J, $H_1 = \{h \in H;\ 0 < h' < \infty$ on $J\}$, $Q = \{h \in H;\ f \circ h \in C_{ap}$ for each $f \in C_{ap}\}$ (where $(f \circ h)(x) = f(h(x))$) and $W = \{f \in D;\ f^2 \in D\}$. For each system $S \subset D$ let $M(S) = \{\varphi \in D;\ \varphi f \in D$ for each $f \in S\}$ and $T(S) = \{h \in H;\ f \circ h \in D$ for each $f \in S\}$.

The systems Q, $M(D)$ and $T(D)$ have been characterized in [1], [3] and [4], respectively; the system $T(W)$ has been investigated in [2]. It is not difficult to show that

(1) $bC_{ap} \subset W \subset L \subset D \cap C_{ap}$,

(2) $M(D) \subset bC_{ap}$, $M(L) = bD$,

(3) $L = \{fg;\ f, g \in W\}$,

(4) $Q = T(bC_{ap})$.

We shall need the following two assertions:

(A_1) Let $h \in Q$, $a \in J$. Then there is a number $\delta > 0$ such that $|h(x) - h(a)|/|x - a|^{\delta} \to 0$ $(x \to a,\ x \in J)$.

(A_2) Let $S \subset D$, $h \in H_1$, $g = h^{-1}$. Then $h \in T(S)$ if and only if $g' \in M(S)$.

The proof of (A_1) can be found in [1]; the proof of (A_2) is very simple.

Let $f_1 \in bD^+ \backslash C_{ap}$, $f_2 \in W^+ \backslash bD$ and let $f_3 = w^2$, where w is a function in W^+ such that $w^3 \notin D$. By (1) - (3) we have $f_1 \in M(L) \backslash M(D)$, $f_2 \in M(W) \backslash M(L)$, $f_3 \in M(C_{ap}) \backslash M(W)$, and it follows easily from (A_2) that the obvious inclusions

$$(5) \qquad T(D) \subset T(L) \subset T(W) \subset T(bC_{ap})$$

are proper. We also see from (2) and (A_2) that there is an $h \in H_1 \backslash T(D)$ such that both functions h' and $(h^{-1})'$ are bounded.

To formulate the main result (A_3) we need the following notation: If f is a function on J and if $x \in J$, then $\overline{D}f(x)$ [$\underline{D}f(x)$] is the upper [lower] derivate of f at x; if $x \in \{0,1\}$, we mean, of course, the corresponding unilateral derivates. If γ is a mapping of J to $[0,\infty]$ and if $a, b \in J$, $a \neq b$, then $\sup(\gamma,a,b)$ means $\sup\{\gamma(x);\ x \in I\}$, where I is the closed interval with endpoints a, b. If $\gamma(x) = \infty$ for some $x \in I$, let var$(\gamma,a,b) = \infty$; otherwise let var(γ,a,b) be the variation of γ on I.

(A_3) Let $h \in H$, $g = h^{-1}$. Let γ be a mapping of J to $[0,\infty]$ such that $\underline{D}g \leqq \gamma \leqq \overline{D}g$. Then we have $h \in T(L)$ if and only if

$$(6) \qquad \limsup \frac{1}{g(x)-g(a)} \int_a^x \sup(\gamma,t,x)dt < \infty$$

$$(x \to a,\ x \in J) \quad \text{for each} \quad a \in J;$$

we have $h \in T(D)$ if and only if

$$(7) \qquad \limsup \frac{1}{g(x)-g(a)} \int_a^x \operatorname{var}(\gamma,t,x)dt < \infty$$

$$(x \to a,\ x \in J) \quad \text{for each} \quad a \in J.$$

The characterization of $T(D)$ by (7) is different from the characterization given in [4].

It follows easily from (6) that the set $\{x \in J;\ \underline{D}h(x) = 0\}$ is finite for each $h \in T(L)$. We see that there are infinitely differentiable functions in $H\backslash T(L)$. According to (A_1), there are convex functions in $H\backslash Q$; by (4) and (5), in $H\backslash T(D)$. It can be proved, however, that $h \in T(D)$ for each convex function $h \in Q \cap H$.

It follows from (6) that $h \in T(L)$, if both h and h^{-1} are Lipschitz functions.

It is easy to prove that $h \in T(D)$ for each $h \in H_1$ such that h' is of bounded variation. It is, however, not difficult to construct a function $h \in H$ such that h'' is continuous and $h' > 0$ on $(0,1]$.

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