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MULTIPLIERS OF NONNEGATIVE DERIVATIVES

Introduction. Throughout this note the word function means a finite real function, i.e. a mapping to $R = (-\infty,\infty)$. Let Φ be a class of functions on a set $J \neq \emptyset$. By $M(\Phi)$ we denote the system of all functions f on J such that $f\varphi \in \Phi$ for each $\varphi \in \Phi$. The elements of $M(\Phi)$ are called multipliers of Φ. The description of $M(\Phi)$ may be trivial; if, e.g., Φ is closed under multiplication and if the function $\varphi(x) = 1$ $(x \in J)$ belongs to Φ, then, obviously, $M(\Phi) = \Phi$. In particular, $M(M(\Phi)) = M(\Phi)$ for any Φ. If, however, Φ "behaves badly" with respect to multiplication, then the investigation of $M(\Phi)$ may lead to some interesting results. Let $J = [0,1]$, let D be the class of all finite derivatives on J and let SD be the class of all summable (= Lebesgue integrable) functions in D. For each class Φ of functions on J let Φ^+ be the class of all nonnegative functions in Φ. The systems $M(D)$ and $M(SD)$ have been characterized in [1] and [2] (see also [3] and [4]). It is natural to investigate $M(D^+)$. Actually, we shall investigate the system $\mathcal{M}$ of all functions f on

J such that $f\varphi \in D$ for each $\varphi \in D^+$; it is easy to see that $M(D^+) = \mathfrak{M}^+$. Some properties of $\mathfrak{M}$ have been stated without proof in [4].

1. Basic properties of $\mathfrak{M}$

Notation. Let C_{ap} be the system of all functions approximately continuous on the interval $J = [0,1]$ and let bC_{ap} be the system of all bounded functions in C_{ap}. Integrals are Lebesgue integrals.

1.1. Lemma. Let f be a function such that $fg \in D$ for each $g \in D^+$ for which $g(0) = 0$. Then

$$\limsup_{x\to 0+} |f(x)| < \infty .$$

Proof. Let, e.g., $\limsup_{x\to 0+} f(x) = \infty$. There are $a_0, a_1, \ldots \in (0,1)$ such that $2a_n < a_{n-1}$ and $f(a_n) > n$ for $n = 1,2,\ldots$. It is easy to see that there is a function F such that $F' = f$ on $(0,1]$. It follows that there are $b_n \in (a_n, 2a_n)$ such that $F(b_n) - F(a_n) > n(b_n - a_n)$. Let g be a nonnegative function continuous on $(0,1]$ such that $g = a_n/(n(b_n - a_n))$ on $[a_n, b_n]$ and $\int_{a_n}^{a_{n-1}} g < 2a_n/n$. Set $g(0) = 0$. If $a_n < x \leqq a_{n-1}$, then $x^{-1} \int_0^x g \leqq a_n^{-1} \int_0^{a_{n-1}} g < 4/n$ so that $g \in D^+$. By assumption there is a function Q such that $Q' = fg$ on J and

$Q(0) = 0$. Obviously $Q'^{+}(0) = 0$ so that $(Q(b_n) - Q(a_n))/a_n = (Q(b_n)/b_n) \cdot (b_n/a_n) - Q(a_n)/a_n \to 0$. However, $Q(b_n) - Q(a_n) = (a_n/(n(b_n - a_n))) \cdot (F(b_n) - F(a_n)) > a_n$ $(n = 1,2,\ldots)$ which is a contradiction.

1.2. Lemma. Let g be a nonnegative measurable function on J such that $x^{-1}\int_0^x g \to 0$ $(x \to 0+)$. Then $\lim \mathrm{ap}_{x\to 0+}\, g(x) = 0$.

(The proof is left to the reader.)

1.3. Lemma. Let f be a function such that $f \in D$ and $f^2 \in D$. Then $f \in C_{ap}$.

Proof. Let $a \in J$. Obviously $(f - f(a))^2 \in D$. It follows easily from 1.2 that f is approximately continuous at a with respect to J. Hence $f \in C_{ap}$.

1.4. Theorem. $\mathcal{M} \subset bC_{ap}$.

Proof. Let $f \in \mathcal{M}$. It is obvious that $f \in D$ and it follows easily from 1.1 that f is bounded. Thus, there is a $c \in R$ such that $f - c \in D^+$. Hence $f \cdot (f - c) \in D$, $f^2 \in D$. Now we apply 1.3.

1.5. Theorem. Let E be the vector space generated by D^+. Then $M(E) = \mathcal{M}$.

Proof. It is easy to see that $E = \{g_1 - g_2; g_1, g_2 \in D^+\}$. Let $f \in \mathcal{M}$ and $g \in D^+$. By 1.4 there is a $c \in R$ such that

$|f| \leq c$ on J. Then $2fg = (c+f)g - (c-f)g \in E$. It follows that $\mathfrak{M} \subset M(E)$. Obviously $M(E) \subset \mathfrak{M}$.

<u>1.6. Lemma</u>. Let g, $f_n \in D$, $\epsilon_n \in (0,\infty)$ $(n = 1,2,\ldots)$, $\epsilon_n \to 0$. Let f be a function on J and let $|f_n - f| \leq \epsilon_n g$ on J for each n. Then $f \in D$.

<u>Proof</u>. Let G, F_n be functions such that $F_n(0) = 0$ and that $G' = g$, $F_n' = f_n$ on J. It is easy to see that there is a function F such that $F_n \to F$ on J. We have $|F(y) - F(x) - (y-x)f(x)| \leq |F_n(y) - F_n(x) - (y-x)f_n(x)| + \epsilon_n |G(y) - G(x)| + |y-x| \cdot |f_n(x) - f(x)|$ $(n = 1,2,\ldots,x,y \in J)$. Hence $F' = f$ on J.

<u>1.7. Theorem</u>. $\mathfrak{M}$ is closed under uniform convergence.

(This follows easily from 1.6.)

<u>Remark</u>. Every function with a continuous derivative on J belongs to $M(D)$, all the more to $\mathfrak{M}$. It follows from 1.7 that each function continuous on J belongs to $\mathfrak{M}$ (which is easy to prove directly).

<u>1.8. Theorem</u>. Let φ be a function continuous on R and let $f \in \mathfrak{M}$. Then the composite function $\varphi \circ f$ belongs to $\mathfrak{M}$.

<u>Proof</u>. By 1.4 there is a compact interval K such that $f(J) \subset K$. There are polynomials $P_1, P_2, \ldots$ such that $P_n \to \varphi$ uniformly on K. The system $\mathfrak{M}$ is a vector

space containing constant functions. It follows from 1.5 that $\mathfrak{M}$ is closed under multiplication. Hence $P_n \circ f \in \mathfrak{M}$ for each n. Obviously $P_n \circ f \to \varphi \circ f$ uniformly. Now we apply 1.7.

2. Characterization of $\mathfrak{M}$

Notation. Let $N = \{1,2,\dots\}$. For each set $S \subset R$ let $|S|$ be its outer Lebesgue measure. If f is a bounded nonnegative function on an interval $I = [a,b]$ and if $r \in N$, we set

$$A(r,I,f) = A(r,a,b,f) = r^{-1}\sum_{k=1}^{r} \sup f([x_{k-1},x_k]),$$
$$\text{where } x_k = a + k(b-a)/r ,$$

and

$$B(r,I,f) = B(r,a,b,f) =$$
$$\inf\{\sum_{k=1}^{r}(y_k - y_{k-1})\sup f([y_{k-1},y_k]);\ a = y_0 < y_1 < \dots < y_r = b\}.$$

2.1. Lemma. Let $a,b,c \in R$, $a < b < c$. Let f be a bounded nonnegative function on $[a,b]$, let g be a bounded nonnegative function on $[a,c]$ and let $r,s \in N$. Then

$$B(r,a,b,f) \leqq (b-a)A(r,a,b,f) ,$$

$$B(r+1,a,b,f) \leqq B(r,a,b,f),\ B(r,a,b,g) \leqq B(r,a,c,g),$$

$$B(r+s,a,c,g) \leqq B(r,a,b,g) + B(s,b,c,g) .$$

(The proof is left to the reader.)

2.2. Lemma. Let $r, s \in N$, $M \in R$. Let I be a compact interval and let f be a function such that $0 \leqq f \leqq M$ on I. Then

$$A(r,I,f) \leqq |I|^{-1} B(s,I,f) + M(s-1)/r \quad (1)$$

Proof. Let $I = [a,b]$, $a = y_0 < y_1 < \cdots < y_s = b$. Set $x_k = a + k|I|/r$, $K = \{k; (x_{k-1}, x_k) \cap \{y_1, \ldots, y_{s-1}\} = \emptyset\}$, $\alpha_k = \sup f([x_{k-1}, x_k])$, $\beta_j = \sup f([y_{j-1}, y_j])$. It is easy to see that $\Sigma_{k \in K}(x_k - x_{k-1})\alpha_k \leqq \Sigma_{j=1}^{s} (y_j - y_{j-1})\beta_j$. Hence $|I| A(r,I,f) = |I| r^{-1} \Sigma_{k=1}^{r} \alpha_k \leqq \Sigma_{j=1}^{s} (y_j - y_{j-1})\beta_j + (s-1)M|I|r^{-1}$ from which (1) follows at once.

2.3. Lemma. Let f be a bounded nonnegative function on J. Then the following properties are equivalent:

i) $2^n B(r, 2^{-n}, 2^{-n+1}, f) \to 0$

ii) $x^{-1} B(r,0,x,f) \to 0$

iii) $A(r,0,x,f) \to 0$

iv) $A(r,0,1/n,f) \to 0$

$(n, r \in N;\ n, r \to \infty,\ x \to 0+)$.

Proof. Suppose that i) holds. Let $M = \sup f(J)$ and let $\varepsilon \in (0,\infty)$. There are $s, n_0 \in N$ such that $2^{k+2} B(s, 2^{-k}, 2^{-k+1}, f) < \varepsilon$ for each $k \in N \cap (n_0, \infty)$. Let $0 < x < 2^{-n_0}$. Choose $n, q \in N$ such that $2^{-n-1} \leqq x < 2^{-n}$ and $2^{q-2} \varepsilon > M$. Obviously $n \geqq n_0$. By 2.1 we have

$B(1+qs,0,x,f) \leqq B(1,0,2^{-n-q},f) + B(s,2^{-n-q},2^{-n-q+1},f) + \cdots +$
$B(s,2^{-n-1},2^{-n},f) \leqq M \cdot 2^{-n-q} + \varepsilon(2^{-n-q-2} + \cdots + 2^{-n-3}) \leqq$
$\varepsilon \cdot 2^{-n-2} + \varepsilon \cdot 2^{-n-2} \leqq \varepsilon x$. This proves ii). If ii) holds, then iii) holds by 2.2; iv) is an obvious consequence of iii). From the inequalities $2^n B(r,2^{-n},2^{-n+1},f) \leqq 2 \cdot 2^{n-1} B(r,0,2^{-n+1},f) \leqq 2A(r,0,2^{-n+1},f)$ we see that iv) implies i).

2.4. Lemma. Let f be a summable derivative on an interval $I = [a,b]$ and let T be a number less than $\sup\{|f(x)|;\ x \in I\}$. Then there is a function g piecewise linear on I such that $g(a) = g(b) = \int_I g = 0$, $\int_I |g| = 2|I|$ and

$$T|I| < \int_I (fg + |f|) .$$

Proof. We may suppose that $\sup\{|f(x)|;\ x \in I\} = \sup f(I)$. Choose an $\varepsilon \in (0,\infty)$ such that the number $V = T + 3\varepsilon$ is less than $\sup f(I)$. There is an $\eta \in (0,\infty)$ such that

$$3\eta \int_I |f| < \varepsilon|I|(|I| - 3\eta) . \tag{2}$$

Since f is a Darboux function, there is a $c \in (a,b)$ such that $f(c) > V$. There is a $d \in (c,b)$ such that $\int_c^d f > V(d-c)$ and that $d - c < \eta$. There is a $\delta \in (0,\eta)$ such that $a < c - \delta$, $d + \delta < b$, $V(d-c) > (V-\varepsilon)(d-c+\delta)$ and that $\int_{c-\delta}^{c} |f| + \int_d^{d+\delta} |f| < \varepsilon(d-c)$. Let $\alpha = |I|/(d-c+\delta)$.

Let g_1 be a function on I such that $g_1 = 0$ on $[a, c-\delta] \cup [d+\delta, b]$, $g_1 = \alpha$ on $[c,d]$ and that g_1 is linear on $[c-\delta, c]$ and on $[d, d+\delta]$. Then $\int_I g_1 = \alpha(d-c+\delta) = |I|$. Since $|\int_{c-\delta}^{c} fg_1 + \int_d^{d+\delta} fg_1| <$ $\alpha\varepsilon(d-c) < \varepsilon|I|$ and $\int_c^d fg_1 = \alpha \int_c^d f > \alpha V(d-c) =$ $|I|V(d-c)/(d-c+\delta) > |I|(V-\varepsilon)$, we have $\int_I fg_1 > |I|(V-2\varepsilon)$.

Let $P = I \setminus (c-\delta, d+\delta)$, $\beta = |I|/(|I| - 3\eta)$. Since $|P| > |I| - 3\eta$, we have $\beta|P| > |I|$. It follows that there is a piecewise linear function g_2 on I such that $g_2 = 0$ on $\{a,b\} \cup [c-\delta, d+\delta]$, $0 \leqq g_2 \leqq \beta$ on I and $\int_I g_2 = |I|$. Therefore (see (2)) $\int_I fg_2 \leqq \beta \int_I |f| =$ $(1 + 3\eta/(|I| - 3\eta)) \int_I |f| < \int_I |f| + \varepsilon|I|$. Since $\int_I f\cdot(g_1 - g_2) > |I|(V-2\varepsilon) - \int_I |f| - \varepsilon|I| = |I|T - \int_I |f|$, we may choose $g = g_1 - g_2$.

2.5. Lemma. Let $f \in \mathfrak{M}$, $f(0) = 0$. Then

$$A(r, 2^{-n}, 2^{-n+1}, |f|) \to 0 \quad (r, n \in N;\ r, n \to \infty).$$

Proof. According to 1.4, f is bounded. Let $r_1, r_2, \ldots \in N$, $r_n \to \infty$. Set $z_n = 2^{-n}$. Fix an $n \in N$ and set $x_k = z_n(1 + k/r_n)$ $(k = 0, \ldots, r_n)$, $I_k = [x_{k-1}, x_k]$, $\sigma_k = \sup\{|f(x)|;\ x \in I_k\}$ $(k = 1, \ldots, r_n)$. It follows from 2.4 that there is a function g_n piecewise linear on J

such that $g_n = 0$ on $[0, z_n]$ and on $[2z_n, 1]$, $\int_{I_k} g_n = g_n(x_{k-1}) = g_n(x_k) = 0$, $\int_{I_k} |g_n| = 2z_n/r_n$ and $(\sigma_k - \frac{1}{n}) z_n/r_n < \int_{I_k} (fg_n + |f|)$ for $k = 1, \ldots, r_n$. Then

$$(3) \qquad A(r_n, z_n, 2z_n, |f|) < \frac{1}{n} + z_n^{-1} \int_{z_n}^{2z_n} (fg_n + |f|) \ .$$

Set $g = \sum_{n=1}^{\infty} g_n$. Let G be a function on J such that $G = \sum_{n=1}^{\infty} |g_n|$ on $(0,1]$ and $G(0) = 2$. It is easy to see that $g, G \in D$; obviously $G \pm g \in D^+$. Since $2g = (G+g) - (G-g)$, we have $fg \in D$. Since $f \in bC_{ap}$, we have also $|f| \in D$. Hence

$$z_n^{-1} \int_{z_n}^{2z_n} (fg + |f|) \to 0 \qquad (n \to \infty) \ .$$

This together with (3) easily implies our assertion.

<u>2.6. Lemma</u>. Let f be a bounded nonnegative measurable function on J such that $x^{-1} B(r, 0, x, f) \to 0$ $(x \to 0+,\ r \in N,\ r \to \infty)$. Let $g \in D^+$. Then

$$x^{-1} \int_0^x fg \to 0 \qquad (x \to 0+) \ .$$

<u>Proof</u>. Let $S = \sup f(J)$ and let $\varepsilon \in (0,\infty)$. There is a $\delta \in (0,1)$ and an $r \in N$ such that $2g(0)B(r,0,x,f) < \varepsilon x$ for each $x \in (0,\delta)$. Set $\alpha = \varepsilon/(4(S+1)r)$. There is an $\eta \in (0,\delta)$ such that $|\int_0^x (g - g(0))| < \alpha x$ for each $x \in (0,\eta)$. Choose such an x. There are x_j such that

$0 = x_0 < x_1 < \ldots < x_r = x$ and that $2g(0)\sum_{k=1}^{r} \sigma_k |I_k| < \varepsilon x$, where $I_k = [x_{k-1}, x_k]$ and $\sigma_k = \sup f(I_k)$. Obviously $|\int_{I_k} (g - g(0))| < \alpha(x_{k-1} + x_k) < 2\alpha x$, $\int_{I_k} g < 2\alpha x + g(0)|I_k|$, $\int_{I_k} fg \leqq 2\alpha S x + g(0)\sigma_k |I_k|$ for each k. Therefore $\int_0^x fg \leqq 2\alpha r S x + g(0)\sum_{k=1}^{r} \sigma_k |I_k| < \varepsilon x$. This completes the proof.

<u>2.7. Theorem</u>. Let f be a bounded measurable function on J. Then the following properties a) - d) are equivalent:

a) $f \in \mathfrak{M}$

b) $2^n B(r, x + 2^{-n}, x + 2^{-n+1}, |f - f(x)|) \to 0$ for each $x \in [0,1)$ and $2^n B(r, x - 2^{-n+1}, x - 2^{-n}, |f - f(x)|) \to 0$ for each $x \in (0,1]$

c) $(y - x)^{-1} B(r, x, y, |f - f(x)|) \to 0$ for each $x \in [0,1)$ and $(x - z)^{-1} B(r, z, x, |f - f(x)|) \to 0$ for each $x \in (0,1]$

d) $A(r, x, x + \frac{1}{n}, |f - f(x)|) \to 0$ for each $x \in [0,1)$ and $A(r, x - \frac{1}{n}, x, |f - f(x)|) \to 0$ for each $x \in (0,1]$ $(n, r \in N;\ n, r \to \infty,\ y \to x+,\ z \to x-)$.

<u>Proof</u>. If $f \in \mathfrak{M}$, then b) holds by 2.5 (see also 2.1). According to 2.3, conditions b) - d) are equivalent. Now suppose that c) holds. Let $g \in D^+$ and let $x \in J$. By 2.6 we have $(y - x)^{-1} \int_x^y (f - f(x)) \cdot g \to 0$ so that

$(y-x)^{-1} \int_x^y fg \to f(x)g(x)$ $(y \to x,\ y \in J)$. This shows that $fg \in D$ and that $f \in \mathcal{M}$ which completes the proof.

3. Points of discontinuity of functions in $\mathcal{M}$

3.1. Theorem. Let $f \in \mathcal{M}$. Then f is Riemann integrable.

Proof. It follows from 1.4 that f is bounded. For each $x \in J$ let

$$\omega(x) = \lim_{h\to 0+} \sup\{|f(t)-f(x)|;\ |t-x| < h,\ t \in J\} .$$

Let $\alpha \in (0,\infty)$, $T = \{x \in J;\ \omega(x) > 2\alpha\}$. It suffices to prove that $|T| = 0$. For each $x \in J$ set $\varphi(x) = |T \cap (0,x)|$. Choose an $x \in [0,1)$ and an $\varepsilon \in (0,\infty)$. By 2.7 there is an $r \in N$ and a $\delta \in (0,\infty)$ such that $B(r,x,y,|f-f(x)|) < \varepsilon\alpha(y-x)$ for each $y \in (x,x+\delta)$. Choose such a y. There are x_j such that $x = x_0 < x_1 < \cdots < x_r = y$ and that $\Sigma_{k=1}^{r} \sigma_k(x_k - x_{k-1}) < \varepsilon\alpha(y-x)$, where $\sigma_k = \sup\{|f(t)-f(x)|;\ x_{k-1} \leqq t \leqq x_k\}$. Let

$$K = \{k;\ T \cap (x_{k-1},x_k) \neq \emptyset\} .$$

Obviously $\varphi(y) - \varphi(x) = |T \cap (x,y)| \leqq \Sigma_{k\in K} (x_k - x_{k-1})$. If $\sigma_k < \alpha$ and $t \in (x_{k-1},x_k)$, then for each $v \in (x_{k-1},x_k)$ we have $|f(v)-f(t)| < 2\alpha$ so that $\omega(t) \leqq 2\alpha$, $k \notin K$. Hence $\varphi(y) - \varphi(x) \leqq \Sigma_{k\in K} \sigma_k\alpha^{-1}(x_k - x_{k-1}) < \varepsilon(y-x)$, $\varphi'^{+}(x) = 0$. Similarly can be proved that $\varphi'^{-}(x) = 0$

for each $x \in (0,1]$. It follows that φ is constant which completes the proof.

Notation. For each function f on J let Δ_f be the set of all points of discontinuity of f. For each set $S \subset R$ let $\mathrm{cl}\, S$ be its closure.

Remark. If $f \in \mathcal{M}$, then, by 3.1, $|\Delta_f| = 0$. Now we shall construct a function $f \in \mathcal{M}$ such that the set Δ_f is perfect and a function $g \in \mathcal{M}$ such that $\Delta_g \cap I$ is uncountable for each interval $I \subset J$.

3.2. Construction of f. Let $\mathfrak{M}_0$ be the set whose only element is the interval J. If $\mathfrak{M}_n$ is a system of disjoint closed subintervals of J, let $\mathfrak{M}_{n+1}$ be the system of all intervals $[a,(2a+b)/3]$ and $[(a+2b)/3,b]$, where $[a,b] \in \mathfrak{M}_n$. In this way we define, by induction, $\mathfrak{M}_n$ for $n = 0,1,\ldots$. Let $\mathfrak{P}_n$ be the system of all intervals $((2a+b)/3,\ (a+2b)/3)$, where $[a,b] \in \mathfrak{M}_{n-1}$ $(n = 1,2,\ldots)$. For each $I = (a,b) \in \mathfrak{P}_n$ define a function λ_I as follows: Set $c = (a+b)/2$, $\delta = 1/(2\cdot 9^n)$, $\alpha = c-\delta$, $\beta = c+\delta$. Let $\lambda_I = 0$ on $(a,\alpha] \cup [\beta,b)$, $\lambda_I(c) = 1$ and let λ_I be linear on $[\alpha,c]$ and on $[c,\beta]$. Since $\beta-\alpha = (b-a)/3^n$, we have $\lambda_I = 0$ on $(a,(2a+b)/3] \cup [(a+2b)/3,b)$. Now define a function f setting $f = \lambda_I$ on $I \in \bigcup_{n=1}^{\infty} \mathfrak{P}_n$ and $f = 0$ elsewhere on J.

It is easy to see that Δ_f is the Cantor set.

3.3. Lemma. Let $I \in \mathfrak{P}_n$. Then $B(3, \mathrm{cl}\, I, f) \leqq 9^{-n}$.

(Obvious.)

3.4. Lemma. Let $L \in \mathfrak{M}_n$ and let $k \in N$. Then

$$B(2^{k+2} - 3, L, f) \leqq |L|\left(\frac{|L|}{7} + \left(\frac{2}{3}\right)^k\right) . \tag{4}$$

Proof. The number of elements of $\mathfrak{P}_{n+j}$ contained in L is 2^{j-1} $(j = 1, \ldots, k)$ and the number of elements of $\mathfrak{M}_{n+k}$ contained in L is 2^k. Since $3(1 + \cdots + 2^{k-1}) + 2^k = 4 \cdot 2^k - 3$, we have (see 2.1 and 3.3) $B(2^{k+2} - 3, L, f) \leqq$ $\sum_{j=1}^{k} \sum_{I \in \mathfrak{P}_{n+j}} B(3, \mathrm{cl}\, I, f) + \sum_{I \in \mathfrak{M}_{n+k}} B(1, I, f) \leqq$ $\sum_{j=1}^{k} 2^{j-1}/9^{n+j} + 2^k/3^{n+k} < 9^{-n}/7 + (2/3)^k/3^n$ which proves (4).

3.5. Lemma. Let C be the Cantor set. Let L be a closed subinterval of J such that $L \cap C \neq \emptyset$ and let k be a natural number. Then

$$B(2^{k+2}, L, f) \leqq |L|(11|L| + 3(2/3)^k) .$$

Proof. We may suppose that $|L| < 1/3$. There is an $n \in N$ such that $3^{-n-1} \leqq |L| < 3^{-n}$. Set $h = 3^{-n}$. There is an integer j such that $L \subset ((j-1)h, (j+1)h)$. Since $L \cap C \neq \emptyset$, we have either $[(j-1)h, jh] \in \mathfrak{M}_n$ or $[jh, (j+1)h] \in \mathfrak{M}_n$. Let, e.g., $[(j-1)h, jh] \in \mathfrak{M}_n$. Then either $(jh, (j+1)h) \in \mathfrak{P}_n$ or $f = 0$ on $[jh, (j+1)h]$ so that, by 3.3 and 3.4, $B(2^{k+2}, L, f) \leqq h(\frac{h}{7} + (\frac{2}{3})^k) + h^2$. Since

$h \leqq 3|L|$, we have $B(2^{k+2},L,f) \leqq |L|((72/7)|L| + 3(2/3)^k)$ which proves our assertion.

3.6. Theorem. $f \in \mathcal{M}$.

Proof. Let $x \in J$. If $x \notin C$, then 2.7, d) follows from the continuity of f at x. If $x \in C$, then 2.7, c) follows from 3.5.

3.7. Theorem. Let f be as in 3.2. Extend f setting $f(x) = 0$ for $x < 0$ and $x > 1$. Let $x_n \in (0,1)$ and let the set $\{x_1,x_2,...\}$ be dense in J. For each $x \in J$ set $g(x) = \sum_{n=1}^{\infty} 4^{-n} f(x - x_n)$. Then $g \in \mathcal{M}$ and $\Delta_g \cap I$ is uncountable for each interval $I \subset J$.

Proof. Let I be an open interval, $I \subset J$. There is an n such that $x_n \in I$. Let m be the smallest natural number such that $x_n - x_m \in C$. (Obviously $m \leqq n$.) Since C is closed, there is an open interval $I_1 \subset I$ such that $x_n \in I_1$ and that $x - x_k \notin C$ for $x \in I_1$ and $k = 1,...,m-1$. Since $x_n - x_m \in C$ and since C is perfect, the set $S = \{x \in I_1; x - x_m \in C\}$ is uncountable. Set $\alpha(x) = \sum_{k<m} 4^{-k} f(x - x_k)$, $\beta(x) = 4^{-m} f(x - x_m)$, $\gamma(x) = \sum_{k>m} \cdots$. Let $s \in S$. It is easy to see that α is continuous at s, $\limsup_{x\to s} \beta(x) = 4^{-m}$, $\liminf_{x\to s} \beta(x) = 0$, $|\gamma(x)| \leqq 1/(3\cdot 4^m)$ for each x. This easily implies that $g = \alpha + \beta + \gamma$ is not continuous at s. It follows from 3.6 and 1.7 that $g \in \mathcal{M}$.

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