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SOME PROPERTIES OF MULTIPLIERS OF SUMMABLE DERIVATIVES

Introduction. Let $J = [0,1]$. For every class Φ of functions on J let $M(\Phi)$ be the system of all functions f on J such that $f\varphi \in \Phi$ for each $\varphi \in \Phi$. The elements of $M(\Phi)$ are called multipliers of Φ. R.J. Fleissner posed in [1] the problem of characterization of the system $M(SD)$, where SD is the class of all summable (= Lebesgue integrable) derivatives. This problem has been solved in [2]. In this note we prove that the set of points of discontinuity of a function in $M(SD)$ is "small" (in particular, countable and nowhere dense) and that some continuous functions in $M(SD)$ are nowhere differentiable.

Notation. The word function means a mapping to $(-\infty,\infty)$. If f is a function on an interval $[a,b]$ and if n is a natural number, then $v(n,a,b,f)$ denotes the least upper bound of all sums $\sum_{k=1}^{n} |f(y_k) - f(x_k)|$, where $a \leq x_1 < y_1 \leq \dots \leq x_n < y_n \leq b$. Let V be the set of all functions f on J such that

$$\limsup_{n\to\infty} v(n, x+\frac{1}{n}, x+\frac{2}{n}, f) < \infty \quad \text{for each} \quad x \in [0,1)$$

and

$$\limsup_{n\to\infty} v(n, x-\tfrac{2}{n}, x-\tfrac{1}{n}, f) < \infty \quad \text{for each} \quad x \in (0,1].$$

If f is a function on J and if $x \in J$, we set

$$\omega(x,f) = \lim_{h\to 0+}(\sup\{|f(t)-f(x)|;\ t \in J,\ |t-x| < h\}).$$

Remark. It is obvious that f is continuous at x (with respect to J) if and only if $\omega(x,f) = 0$. Hence $\{x; \omega(x,f) > 0\}$ is the set of all points of discontinuity of f.

1. Theorem. A function belongs to $M(SD)$ if and only if it is a derivative belonging to V.

Proof. Let W be the system defined in section 6 of [2]. It is easy to prove that $W = V$. Now we apply Theorem 8 of [2].

2. Lemma. Let f be a Darboux function on J and let n be a natural number. Let $a, b, x \in J$, $a < x < b$. Then $v(n,a,b,f) \geq n\omega(x,f)$.

Proof. We may suppose that $\omega(x,f) > 0$. Let $\varepsilon \in (0, \omega(x,f))$. There is a $y_1 \in (a,b)$ such that $|f(y_1) - f(x)| > \varepsilon$. Since f is a Darboux function, there is an $x_1 \in (a,b)$ such that $0 < |x - x_1| < |x - y_1|$ and that $|f(y_1) - f(x_1)| > \varepsilon$. There is a $y_2 \in (a,b)$ such that $|x - y_2| < |x - x_1|$ and $|f(y_2) - f(x)| > \varepsilon$ etc.

In this way we construct disjoint intervals with endpoints $x_j, y_j \in (a,b)$ such that $|f(y_j) - f(x_j)| > \varepsilon$ for $j = 1,\dots,n$. Hence $v(n,a,b,f) > n\varepsilon$ which proves our assertion.

3. Lemma. Let f be a Darboux function on J and let $x \in [0,1)$. Then

$$(1) \quad \limsup_{y \to x^+} (y-x)^{-1}\omega(y,f) \leq \limsup_{n\to\infty} v(n, x+\tfrac{1}{n}, x+\tfrac{2}{n}, f).$$

Proof. Let A be the right-hand side of the inequality (1). We may suppose that $A < \infty$. Let $B \in (A,\infty)$. There is a $p \in (1,\infty)$ such that $v(n,x+\frac{1}{n},x+\frac{2}{n},f) < B$ for each $n > p$. Let $y \in (x, x+\frac{1}{p})$ and let n be an integer such that $1/(y-x) < n < 2/(y-x)$. Obviously $x+\frac{1}{n} < y < x+\frac{2}{n}$ and $n > p$. Hence, by Lemma 2, $(y-x)^{-1}\omega(y,f) \leq n\omega(y,f) \leq v(n,x+\frac{1}{n},x+\frac{2}{n},f) < B$ which proves (1).

4. Lemma. Let $f \in V$ and let f be a Darboux function. Let $\emptyset \neq T \subset \{x; \omega(x,f) > 0\}$. Then T has a left-isolated point.

Proof. Suppose that no point of T is left-isolated. Choose a $b_0 \in T$ and set $a_0 = b_0 - 1$. Suppose that n is a positive integer and that a_{n-1}, b_{n-1} are numbers such that $a_{n-1} < b_{n-1} \in T$. It is easy to see that there is a $b_n \in T$ and a number a_n such that $a_{n-1} < a_n < b_n < b_{n-1}$ and $n(b_n - a_n) < \omega(b_n, f)$. In this way we construct sequences

$a_0, a_1, \ldots$ and $b_0, b_1, \ldots$. Let $b_n \to b$. Obviously $a_n < b < b_n$ and $n(b_n - b) < n(b_n - a_n) < \omega(b_n, f)$ for each n. This contradicts Lemma 3.

5. Proposition. Let $f \in V$ and let f be a Darboux function. Let $\emptyset \neq T \subset \{x; \omega(x,f) > 0\}$. Then T has an isolated point.

Proof. Suppose that no point of T is isolated. Let L be the set of all left-isolated points of T. By Lemma 4 there is an $a_0 \in L$. Set $b_0 = a_0 + 1$. Suppose that n is a positive integer and that a_{n-1}, b_{n-1} are numbers such that $b_{n-1} > a_{n-1} \in L$. Set $T_0 = (a_{n-1}, b_{n-1}) \cap T$. By assumption $T_0 \neq \emptyset$. By Lemma 4 T_0 has a left-isolated point, say a_n; it is easy to see that $a_n \in L$. There is a $b_n \in (a_n, b_{n-1})$ such that $n(b_n - a_n) < \omega(a_n, f)$. In this way we construct sequences $a_0, a_1, \ldots, b_0, b_1, \ldots$. Let $a_n \to a$. Obviously $a_n < a < b_n$ and $n(a - a_n) < n(b_n - a_n) < \omega(a_n, f)$ for each n. This contradicts the "symmetrical version" of Lemma 3.

6. Theorem. Let $f \in M(SD)$ and let $\varepsilon \in (0, \infty)$. Then the set $\{x \in J;\ \omega(x,f) > \varepsilon\}$ is finite and each nonempty subset of $\{x;\ \omega(x,f) > 0\}$ has an isolated point.

Proof. According to Theorem 1 f is a Darboux function belonging to V. It follows from Lemma 3 that $\omega(y,f) \to 0$ $(y \to x,\ y \in J)$ for each $x \in J$. This easily implies that the set $\{x;\ \omega(x,f) > \varepsilon\}$ is finite. The second assertion follows at once from Proposition 5.

Remark. It is obvious that each function monotone on J belongs to V. However, such a function may be discontinuous at each point of a dense set. We see that in Theorem 6 the assumption $f \in M(SD)$ cannot be replaced by the requirement $f \in V$.

7. Lemma. Let $A, B, a_1, a_2, \ldots, b_1, b_2, \ldots$ be positive numbers such that $\sum_{k>n} a_k \leqq Aa_n$, $\sum_{k<n} b_k \leqq Bb_n$ for each n and that $\sup_k a_k b_k < \infty$. Let $f_1, f_2, \ldots$ be functions on J such that $|f_n| \leqq a_n$ on J and that $|f_n(y) - f_n(x)| \leqq b_n|y-x|$, whenever $x, y \in J$ $(n = 1, 2, \ldots)$. Then $\sum_{n=1}^{\infty} f_n \in M(SD)$.

Proof. Let $Q = \sup_k a_k b_k$. Let n be an integer greater than b_1. It is obvious that $\sup_k b_k = \infty$. Let K be the smallest natural number such that $b_K > n$. Let $\alpha, \beta \in J$, $\beta = \alpha + \frac{1}{n}$. Set $\varphi = \sum_{k<K} f_k$, $\psi = \sum_{k\geqq K} f_k$, $f = \varphi + \psi$. It is easy to see that $v(n,\alpha,\beta,\varphi) \leqq \frac{1}{n}\sum_{k<K} b_k \leqq (B+1)b_{K-1}/n \leqq B+1$, $v(n,\alpha,\beta,\psi) \leqq 2n\sum_{k\geqq K} a_k \leqq 2n(A+1)a_K \leqq 2(A+1)Qn/b_K \leqq 2Q(A+1)$ so that $v(n,\alpha,\beta,f) \leqq B+1+2Q(A+1)$. Hence $f \in V$. Since f is continuous, we have $f \in M(SD)$.

8. Lemma. Let $A, B, a_1, a_2, \ldots, b_1, b_2, \ldots$ be positive numbers such that $\sum_{k>n} a_k \leqq Aa_n$, $\sum_{k<n} b_k \leqq Bb_n$ for each n and that $2A + 3B < 1$. Let φ be a 2-periodic function such that $\varphi(x) = |x|$ for $|x| \leqq 1$. Set $f(x) =$

$\sum_{k=1}^{\infty} a_k\varphi(b_k a_k^{-1}x)$. Then for each real x we have $D^+f(x) = \infty$, $D_-f(x) = -\infty$ or $D_+f(x) = -\infty$, $D^-f(x) = \infty$.

Proof. Let $x \in (-\infty,\infty)$ and let n be a natural number. Set $d = a_n/b_n$. There is an integer j such that $|x - dj| \leqq d/2$. Set $y = (j+1)d$, $z = (j-1)d$. Suppose first that j is even. For each k let $\varphi_k(t) = a_k\varphi(b_k a_k^{-1}t)$ $(t \in (-\infty,\infty))$. We have $f(y) - f(x) = \sum_{k<n}(\varphi_k(y) - \varphi_k(x)) + \varphi_n(y) - \varphi_n(x) + \sum_{k>n}\varphi_k(y) - \sum_{k>n}\varphi_k(x)$. It is easy to see that $|\varphi_k(y) - \varphi_k(x)| \leqq b_k(y-x)$ for each k; moreover, $\varphi_n(y) = a_n\varphi(j+1) = a_n$, $0 \leqq \varphi_n(x) \leqq a_n/2$. Since $y - x \leqq \frac{3}{2}d$, we have $a_n/2 = d\,b_n/2 \geqq b_n(y-x)/3$ so that $f(y) - f(x) \geqq -(y-x)\sum_{k<n}b_k + a_n/2 - \sum_{k>n}a_k \geqq -(y-x)Bb_n + a_n/2 - Aa_n \geqq -(y-x)Bb_n + (b_n(y-x)/3)(1-2A) = (y-x)c_n$, where $c_n = b_n(1-2A-3B)/3$. In the same way it can be proved that $f(z) - f(x) \geqq (x-z)c_n$. If j is odd, we proceed similarly. Set $j_n = j$, $y_n = y$, $z_n = z$. Then $z_n < x < y_n$, $z_n \to x$, $y_n \to x$; for j_n even we have

$$\frac{f(y_n) - f(x)}{y_n - x} \geqq c_n, \quad \frac{f(z_n) - f(x)}{z_n - x} \leqq -c_n,$$

for j_n odd we have

$$\frac{f(y_n) - f(x)}{y_n - x} \leqq -c_n, \quad \frac{f(z_n) - f(x)}{z_n - x} \geqq c_n.$$

Obviously $c_n \to \infty$. This completes the proof.

9. Theorem. Let $q \in (6,\infty)$. Let φ be as in Lemma 8. For each $x \in [0,1]$ set $f(x) = \sum_{k=1}^{\infty} q^{-k}\varphi(q^{2k}x)$. Then f is continuous, $f \in M(SD)$ and f is nowhere differentiable.

Proof. We apply 7 and 8 with $a_k = q^{-k}$, $b_k = q^k$, $A = B = 1/(q-1)$.

REFERENCES

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