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On a Decomposition of $C_o(\mathbb{R}^n)$ Functions into Simple Component Pieces

Suppose $f(x)$ is continuous and tends to 0 at infinity. Then f may be expressed in a special way as the sum of a series: $f = \sum_k \pm h_k$, where each h_k is a "hump": $h_k \geq 0$; for some c h_k is non-decreasing on $(-\infty, c)$, non-increasing on $(c, +\infty)$, and $h_k(x) \to 0$ as $|x| \to \infty$. The h_k "fit together": if $h_k h_\ell \neq 0$, then they receive the same sign, and one of them is constant on the support of the other. This means that if f is integrable, then $\int |f(x)|\,dx = \sum_k \int |h_k(x)|\,dx$, and, if f is of bounded variation, then $\operatorname{var}(f) = \sum_k \operatorname{var}(h_k)$. This latter is done in [B,pp 159-178]. The extension to continuous functions was done simply by Humke and O'Malley (personal communication).

What happens in n dimensions? The object is to express $f \in C_o(\mathbb{R}^n)$ (meaning $f \to 0$ at ∞ and is continuous) as $\Sigma \pm b_k$, where each b_k is somehow "simple", and the b_m "fit together" appropriately. I wish to take "simple" to mean that for each y, $\{b > y\}$ is connected if not empty. In one dimension, this means that b is a "hump". In n dimensions, it allows the radial function $f(x) = r(1-r)_+$ $(r = |x|)$ to qualify as "simple". I would like to know whether the space of continuous BV functions can be treated in this way. BV is the collection of "functions" f, locally integrable, with grad f (in the sense of distributions) a vector-valued locally totally finite signed Borel measure.

At present I can only treat the case of Lipschitz functions which tend to 0 at ∞. In this note I will first talk about $C_o(\mathbb{R}^n)$, then about Lipschitz functions in C_o.

A non-linear operation. Let $f \geq 0$ be in C_o and suppose $\{f \neq 0\}$ is connected. Each function in C_o can be split into a (signed) sum of such functions. Let $M = \{x: f(x) = \max f\}$. For an open set $\mathfrak{G}$, let $\mathrm{comp}_x \mathfrak{G}$ denote the component of $\mathfrak{G}$ containing x ($\mathrm{comp}_x \mathfrak{G} = \emptyset$ if $x \notin \mathfrak{G}$). Set

$$Tf(x) = \sup\{y < f(x) : M \cap \mathrm{comp}_x\{f > y\} \neq \emptyset\} ,$$

where $\sup \emptyset = 0$ by convention. In effect, the graph of Tf is formed by cutting "small hills" off the graph of f. Note that Tf is constant on each component of $\{f - Tf \neq 0\}$, and that the modulus of uniform continuity of Tf is no greater than that of f. It is not necessarily true that $\{Tf > y\}$ is connected for each y, because the set M need not be connected. However, $h = Tf$ can be written as a sum of functions which do satisfy the connectivity condition.

<u>Lemma</u>: Let $h \not\equiv 0$, $h \geq 0$ be in $C_o(\mathbb{R}^n)$ with $\{h > 0\}$ connected, and, for each $y > 0$, suppose each component of $\{h > y\}$ meets $M = \{x: h(x) = \max h\}$. Then $h(x) = \sum_{k=1}^{\infty} b_k(x)$, where each b_k is non-negative, in C_o, and $\{b_k > y\}$ is connected (or empty) for each y.

<u>Proof</u>: Let $\#(y)$ denote the number of components of $\{h > y\}$. Since M is compact, $\#(y) < \infty$ for each y. Now $\#(y)$ is non-

decreasing, and $\#(y+0) = \#(y)$, (so that $\#$ is right-continuous on $[0,\max h)$). The first statement is immediate; if the second were false, some component $\mathfrak{G}$ of $\{h>y\}$ would contain at least two components of $\{h>\bar{y}\}$ for each $\bar{y}$, $y<\bar{y}<\max h$. Then there are points $x_1 \neq x_2$ on $\mathfrak{G} \cap M$ such that for any arc γ lying in $\mathfrak{G}$, joining x_1 and x_2 we have $y<h(\gamma(t))$, $0 \leq t \leq 1$ and $h(\gamma(t_{\bar{y}})) \leq \bar{y}$ for some $t_{\bar{y}}$ in $(0,1)$, each $\bar{y} \in (y,\max h)$. Then $h(\gamma(t_o)) \leq y$ for some t_o (a limit point of $t_{\bar{y}}$'s), which is absurd. Let $J = \{y:\#(y)>\#(y-0)\}$. Then since $\#$ is integer valued, J has at most one limit point, namely $\max h$, so J can be expressed as $0<y_1<y_2<\ldots$ ($0<y_1$, by right continuity). Note that $\#(y)=1$ for $0 \leq y<y_1$. Now define $b_o(x) = \min(y_1,h(x))$, and denote the components of $\{h>y_1\}$ by $\mathfrak{G}_{1i}$, $i=1,\ldots,\#(y_1)$. If $y_1<\max h$, set

$$b_{1i}(x) = \begin{cases} \min((h(x)-y_1)^+,y_2) , & x \in \mathfrak{G}_{1i} ; \\ 0 & \text{otherwise} . \end{cases}$$

The definition of b_{ki} , $1 \leq i \leq \#(y_k)$, is similar. By the construction each b_{ki} satisfies the connected condition: $\{b_{ki}>y\}$ is connected if non-empty. It is clear that $h = \sum_k \sum_{i=1}^{\#(y_k)} b_{ki}$. This completes the proof of the lemma.

Now suppose f is continuous and tends to 0 at ∞ . Let $\mathfrak{G}_k$ denote the components of $\{f\neq 0\}$, $\epsilon_k = \pm 1$ denote the sign of f on $\mathfrak{G}_k$, and set $f_k(x) = \epsilon_k f(x)$ if $x \in \mathfrak{G}_k$, $f_k(x)=0$ otherwise. Then $f = \sum_k \epsilon_k f_k$. Let $h_{k1} = Tf_k$, $f_{k1} = f_k-Tf_k$. Note that h_{k1} is constant on each component of $\{f_{k1}\neq 0\}$. It now follows from the uniform continuity of f that if this process is indefinitely repeated - i.e., write

$f_{kl} = \sum_{\ell} \epsilon_{kl\ell} f_{kl\ell}$, $h_{kl\ell} = Tf_{kl\ell}$, $f_{kl\ell 2} = f_{kl\ell} - h_{kl\ell}$, etc., that f can be expressed as $\sum_m \pm b_m$, where each b_m satisfies the conditions of the Lemma. This yields the following theorem.

<u>Theorem</u>: If f is a continuous real-valued function on $\mathbb{R}^n$ and tends to 0 at infinity, then f can be represented by a series $f = \sum_k \epsilon_k b_k$, $\epsilon_n = \pm 1$, in which each b_k is continuous, non-negative, tends to 0 at infinity, and which in addition satisfies: $\{b_k > y\}$ is, for each y , connected or else empty. Moreover, if $b_k b_\ell \neq 0$, then $\epsilon_k = \epsilon_\ell$, and one of b_k, b_ℓ is positive and constant on the support of the other.

(the support of a function g is the closure of $\{g \neq 0\}$).

It is evident that $\int |f(x)|dx = \sum_m \int |b_m(x)|dx$. If we had $\text{supp}(\text{grad } b_i) \cap \text{supp}(\text{grad } b_j) = \emptyset$ for $i \neq j$ it would be nearly evident that the variations would add, where
$\text{var } b \equiv \sup\{\int b \text{ div } \varphi dx$: φ a C^∞ vector field with compact support and $|\varphi| = (\sum_{j=1}^{n} \varphi_j^2)^{1/2} \leq 1\}$. Since the supports of the b_m are merely non-overlapping, some extra conditions on the b_m seem to be needed.

<u>Proposition</u> Suppose $\{f_m\}$ is a sequence of functions in $C_o(\mathbb{R}^n)$ which are equi-Lipschitz continuous (i.e. for some K , $|f_m(u) - f_m(v)| \leq K|u-v|$, all m). If for each i,j $f_i f_j \geq 0$, and, if $i \neq j$, one of f_i, f_j is constant on the support of the other, then, with $f = \sum_m f_m$, if f is locally integrable, then $\text{var } f = \sum_m \text{var } f_m$. As a corollary of the proposition and the theorem, the decomposition described in the latter satisfies

$\int|f(x)|dx = \sum_m \int|b_m(x)|\,dx$ and $\operatorname{var} f = \sum_m \operatorname{var} b_m$.

Proof of the proposition: We have to show that $\operatorname{var} f \geq \Sigma \operatorname{var} f_m$. The idea is to find a smooth vector field φ_m , $|\varphi_m| \leq 1$, with support in $\mathfrak{G}_m$ (defined to be the interior of the support of grad f_m), such that $\int f_m \operatorname{div} \varphi_m dx$ is close to $\operatorname{var} f_m$. The condition on f_i, f_j for $i \neq j$ implies that grad f_i , grad f_j , have non-overlapping supports. Thus if φ_m is as above, then $\int f \operatorname{div} \varphi_m dx = \sum_k \int f_k \operatorname{div} \varphi_m dx = \int f_m \operatorname{div} \varphi_m dx$, because $\operatorname{supp} \varphi_m \cap \bar{\mathfrak{G}}_k = \emptyset$ if $k \neq m$. Thus if $\int f_m \operatorname{div} \varphi_m dx \geq \operatorname{var} f_m - \epsilon/2^m$, we have $\operatorname{var} f \geq \int f \operatorname{div}(\sum_{k=1}^{K} \varphi_k)dx$ (since the φ_k have pairwise disjoint supports)

$$= \sum_{k=1}^{K} \int f_k \operatorname{div} \varphi_k dx \geq \sum_{k=1}^{K} \operatorname{var} f_k - \epsilon .$$

Now we let K tend to infinity; since ϵ is arbitrary we have the desired result. Thus the problem is to show that we can approximate $\operatorname{var} f_k$ at closely as desired by $\int f_k \operatorname{div} \varphi_k dx$, where φ_k has support in $\mathfrak{G}_k$, is a C^∞ vector field and $|\varphi_k| \leq 1$. We may drop the subscript k . Thus, given $\epsilon > 0$,
$\int f \operatorname{div} \psi \, dx \geq \operatorname{var} f - \frac{1}{2}\epsilon$ for some C^∞ vector field ψ with compact support, $|\psi(x)| \leq 1$ for each x . At least this holds when $\operatorname{var} f < \infty$; if $\operatorname{var} f = +\infty$, we make the right hand side $N + \frac{1}{2}\epsilon$ for some large N . We wish to find a vector field φ like ψ , but with support inside $\mathfrak{G}$ = interior of closure of $\{f \neq 0\}$, so that $\int f \operatorname{div} \varphi \, dx \geq \operatorname{var} f - \epsilon$ (or $\geq N$) .

To do this I need a partition of unity related to a Whitney decomposition of $U = \mathfrak{G} \cap \operatorname{Int}(\operatorname{supp} \psi)$. This may be found on pages

167-170 of [S]. The Whitney decomposition of a bounded open set U expresses it as the union of a denumerable collection of non-overlapping closed cubes Q_j, $j \geq 1$, with $1 \leq \mathrm{dist}(Q_j, \partial U)/\mathrm{diam}\, Q_j \leq 4$. Let each cube Q_j be expanded about its center by a factor $\frac{9}{8}$ and call the new cubes Q_j^*. Then $\frac{8}{11} < \mathrm{dist}(Q_j^*, \partial U)/\mathrm{diam}\, Q_j^* < 4$, and each point of U belongs to at most N of the cubes Q_j^*, where N depends only on n. The partition of unity starts with a C^∞ function P, $0 \leq P \leq 1$, which is identically 1 on the cube Q_0 of unit edge centered at 0, and supported in $(9/8)Q_0$. Then P_j^* is defined by $P_j^*(x) = P((x-c_j)/e_j)$, where c_j is the center of Q_j and e_j is its edge length. Finally, $P_j(x) = P_j^*(x)/\sum_1^\infty P_k^*(x)$ gives a partition of unity for U. We need to use: $|\partial P_j/\partial x_i| \leq A(\mathrm{diam}\, Q_j)^{-1}$, where A depends only on n. This follows from the properties of the cubes, the construction of P_j.

Now suppose $\mathrm{var}\, f - \frac{1}{2}\epsilon \leq \int f \,\mathrm{div}\, \psi \, dx$. We have $\int f \,\mathrm{div}\, \psi \, dx = \lim_{\delta \to 0} \int f(\Sigma_\delta P_j)\mathrm{div}\, \psi \, dx$, where $\Sigma_\delta P_j$ denotes the sum over those j (finite in number) such that $\mathrm{dist}(Q_j^*, \partial U) \geq \delta$. Now, $\int f(\Sigma_\delta P_j)\mathrm{div}\, \psi \, dx = \int f \,\mathrm{div}((\Sigma_\delta P_j)\psi)dx - \int f\, \psi \cdot (\Sigma_\delta \mathrm{grad}\, P_j)dx$, since $a \,\mathrm{div}\, b = \mathrm{div}\, ab - \mathrm{grad}\, a \cdot b$. The first term has the desired form. It remains to show that δ can be chosen sufficiently small that the second term on the right-hand side is less than, say, $\epsilon/4$. Since $\Sigma_\delta P_j \equiv 1$ if $\mathrm{dist}(x, \partial U) \geq A\delta$, for some $A > 1$ depending on n (by the geometric properties of the cubes Q_j^*), and because of the estimate on the derivatives of the P_j, $\mathrm{grad}\, \Sigma_\delta P_j = 0$ if $\mathrm{dist}(x, \partial U) > A\delta$, and so

$|\mathrm{grad}\ \Sigma_\delta P_j| \le C\delta^{-1}$, where C depends on n only. Thus since $|\psi| \le 1$ and $|f(x)| \le K\ \mathrm{dist}(x, \partial U)$,

$$(*)\qquad \left|\int f\psi\cdot(\mathrm{grad}\ \Sigma_\delta P_j)dx\right| \le K A \delta C \delta^{-1} \cdot \mathrm{meas}\{x: \delta < \mathrm{dist}(x,\partial U) < A\delta\}$$

which tends to 0 as $\delta \to 0$. Thus if δ_0 is so small that $\left|\int f\ \mathrm{div}\ \psi\ dx - \int f(\Sigma_\delta P_j)\mathrm{div}\ \psi\ dx\right| < \epsilon/4$ for all $\delta < \delta_0$, we can choose δ so small that the right-hand side of $(*)$ is $< \epsilon/4$.

Since this is done for each f_k, and the sum is locally integrable, and sum and integral can be interchanged because of the condition $f_i f_j \ge 0$, the proof of the proposition is complete.

Remark: The estimate $(*)$ shows that if we know a rate of decay for $\mathrm{meas}\{x: \delta < \mathrm{dist}(x,\partial U) < A\delta\}$, we can correspondingly weaken the condition on f. However, improvement in the result is more likely to come from a different approach.

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References

[B] Wm. Beckner, et al, eds., Conference on harmonic analysis in honor of Antoni Zygmund, vol 1, pp 159-178: On the decomposition of $L_1^1(\mathbb{R})$ functions into humps, by Max Jodeit, Jr.

[S] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton, 1970.