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On the classification of set-valued functions

Using a refined Baire classification of multifunctions (i.e. closed set-valued functions), we present here some new results on multifunctions which are similar in spirit to some of the well-known theorems on functions due to Baire, Lebesgue, Hausdorff, Hahn and Banach. These results were developed in connection with a new notion of derivative [3] which is set-valued.

1. Introduction. Given any topological space $(Y,\mathcal{U})$, let $\mathcal{P}(Y)$ denote the space of all closed subsets of Y. Define, for each $U \in \mathcal{U}$,

$$U_* = \{F \in \mathcal{P}(Y): F \subset U\} \quad \text{and} \quad U^* = \{F \in \mathcal{P}(Y): F \cap U \neq \emptyset\}.$$

We shall use the Vietoris topology on the space $\mathcal{P}(Y)$ which is generated by the family $\{U_*: U \in \mathcal{U}\} \cup \{U^*: U \in \mathcal{U}\}$.

Given any other topological space X, let $\mathcal{M}(X,Y)$ denote the space of all multifunctions $\phi: X \to \mathcal{P}(Y)$. We shall assume throughout that X is a perfect space, viz. every open set in X is an F_σ-set. Let $\mathcal{M}^*(X,Y)$ and $\mathcal{M}_k(X,Y)$ denote, further, the spaces of multifunctions in $\mathcal{M}(X,Y)$ whose values are nonempty or compact respectively. When Y is a topological vector space, we use similarly $\mathcal{M}_c(X,Y)$ to denote the space of convex-valued multifunctions in $\mathcal{M}(X,Y)$. We set, further, $\mathcal{M}_{k,c}(X,Y) = \mathcal{M}_k(X,Y) \cap \mathcal{M}_c(X,Y)$, $\mathcal{M}_k^*(X,Y) = \mathcal{M}^*(X,Y) \cap \mathcal{M}_k(X,Y)$, etc.

Next, given any countable ordinal number α, a multifunction $\phi \in \mathcal{M}(X,Y)$ will be said to be of Baire class α if for each open set G in $\mathcal{P}(Y)$ the set $\phi^{-1}(G)$ is of additive class α in X. Let ϕ be said, further, to be of lower or upper

Baire class α if for each open set U in Y the set $\phi^{-1}(U^*)$ or $\phi^{-1}(U_*)$ repsectivley is of additive class α in X.

We shall use $\mathbf{B}_\alpha$ to denote the Baire class α and $\mathbf{LB}_\alpha$ and $\mathbf{UB}_\alpha$ to denote the lower and upper Baire classes α respectively. Thus $\mathbf{LB}_0$ and $\mathbf{UB}_0$ are the well-known classes of lower and upper semicontinuous multifunctions respectively, or briefly LSC and USC multifunctions respectively, and $\mathbf{B}_0$ is simply the class of continuous multifunctions (see e.g. [5, p. 173]).

The classes $\mathbf{LB}_\alpha$ and $\mathbf{UB}_\alpha$ have been considered by Kuratowski in one of his recent papers [6], under a different notation, where many of the known results on LSC and USC multifunctions have been extended to the general classes $\mathbf{LB}_\alpha$ and $\mathbf{UB}_\alpha$. The results presented here are however new even for $\alpha = 0$.

Given any pair of multifunctions $\phi, \psi \in M(X,Y)$, we say $\phi \subset \psi$ provided $\phi(x) \subset \psi(x)$ for each $x \in X$. The multifunctions $\phi \cup \psi$ and $\phi \cap \psi$ are defined in turn by

$$(\phi\cup\psi)(x) = \phi(x) \cup \psi(x), \ (\phi\cap\psi)(x) = \phi(x) \cap \psi(x), \quad x \in X.$$

Next, a sequence of multifunctions $\{\phi_n\}$ in $M(X,Y)$ will be said to converge to a multifunction $\phi \in M(X,Y)$ if the sequence $\{\phi_n(x)\}$ converges to $\phi(x)$ in the Vietoris topology of $P(Y)$ for each $x \in X$. Further, $\{\phi_n\}$ will be called nondecreasing or nonincreasing if $\phi_n \subset \phi_{n+1}$ or $\phi_n \supset \phi_{n+1}$ respectively for each n. The multifunctions $\bigcup_{n=1}^{\infty} \phi_n$ and $\bigcap_{n=1}^{\infty} \phi_n$ are defined in turn by

$$(\bigcup_{n=1}^{\infty} \phi_n)(x) = \{\bigcup_{n=1}^{\infty} \phi_n(x)\}^-, \ (\bigcap_{n=1}^{\infty} \phi_n)(x) = \bigcap_{n=1}^{\infty} \phi_n(x), \ x \in X.$$

Finally, if there exists a finite set of functions $f_i: X \to Y$ $(i=1,2,..,n)$ such that $\phi = \bigcup_{i=1}^{n} f_i$, then ϕ will be called an elementary multifunction and $\{f_i: i = 1,2,...,n\}$ will be called the elements of ϕ.

We state here further a few elementary results on multifunctions. These results have been obtained in [6] in the case when Y is a compact metric space. Unless stated otherwise, we shall always assume α to be an arbitrary countable ordinal number.

It is clear from the definitions that a multifunction $\phi \in \mathbf{B}_0$ iff $\phi \in \mathbf{LB}_0 \cap \mathbf{UB}_0$. For $\alpha > 0$ we have, on the other hand,

1.1 THEOREM. If $\phi \in \mathcal{M}_k(X,Y)$ and Y is second countable, then $\phi \in \mathbf{B}_\alpha$ iff $\phi \in \mathbf{LB}_\alpha \cap \mathbf{UB}_\alpha$.

As regards the mutual relationship between the lower and upper Baire classes, we have

1.2 THEOREM. Suppose $\phi \in \mathcal{M}(X,Y)$ where Y is a perfect space.

(a) If $\phi \in \mathbf{UB}_\alpha$, then $\phi \in \mathbf{LB}_{\alpha+1}$.

(b) If ϕ is compact-valued and in $\mathbf{LB}_\alpha$, and if Y is further normal, then $\phi \in \mathbf{UB}_{\alpha+1}$.

Thus if $\phi \in \mathcal{M}_k(X,Y)$, where Y is a second countable perfect space, and if either (i) $\phi \in \mathbf{UB}_\alpha$ or (ii) Y is normal and $\phi \in \mathbf{LB}_\alpha$, then $\phi \in \mathbf{B}_{\alpha+1}$.

1.3 THEOREM. If two multifunctions $\phi, \psi \in \mathcal{M}(X,Y)$ are both in $\mathbf{LB}_\alpha$ or in $\mathbf{UB}_\alpha$, then so is $\phi \cup \psi$.

Hence if all the elements of some elementary multifunction $\phi \in \mathcal{M}(X,Y)$ are in $\mathbf{B}_\alpha$, it follows from Theorems 1.1 and 1.3 that $\phi \in \mathbf{B}_\alpha$ provided Y is second countable, and this holds for $\alpha = 0$ without any hypothesis on Y.

To avoid further definitions, most of the results are presented here under the hypothesis of separability on Y. In the forthcoming paper we develop, however, a theory which applies to nonseparable Y as well. An application of this theory yields extension of many results on the Baire classification of functions to functions with a nonseparable range. It yeilds also new results on the lower and upper Baire classes of real-valued functions which are defined similar to the Baire's definitions [1] of lower and upper semicontinuous functions.

2. Characterizations of $\mathbf{LB}_\alpha$ and $\mathbf{UB}_\alpha$. We present here characterizations of multifunctions in $\mathbf{LB}_\alpha$ and $\mathbf{UB}_\alpha$ in terms of limits of monotone sequences of multifunctions in $\mathbf{B}_\alpha$. These results are similar to the Baire's characterizations of lower and upper semicontinuous real-valued functions (see [1] or [4, p. 280]). There is however no duality between the classes $\mathbf{LB}_\alpha$ and $\mathbf{UB}_\alpha$, and hence their characterizations are independent of each other.

Let us state first a general result on the Baire class of limits of monotone sequences of multifunctions.

2.1. THEOREM. Let $\{\phi_n\}$ be a sequence of multifunctions in $\mathcal{M}(X,Y)$ and suppose Y is T_1.

(a) If $\{\phi_n\}$ is nondecreasing and $\phi_n \in \mathbf{LB}_\alpha$ for each n, then $\{\phi_n\}$ converges to $\bigcup_{n=1}^{\infty} \phi_n$ which is in $\mathbf{LB}_\alpha$.

(b) If $\{\phi_n\}$ is nonincreasing and ϕ_n is compact-valued and in $\mathbf{UB}_\alpha$ for each n, then $\{\phi_n\}$ converges to $\bigcap_{n=1}^{\infty} \phi_n$ which is in $\mathbf{UB}_\alpha$.

2.2. THEOREM. Suppose Y is a separable metric space, $\phi \in \mathcal{M}^*(X,Y)$ is complete-valued and that $\alpha > 0$. Then $\phi \in \mathbf{LB}_\alpha$ iff it is the limit of a nondecreasing sequence of elementary multifunctions $\{\phi_n\}$ whose elements are in $\mathbf{B}_\alpha$.

The above theorem is easily seen to contain the selection theorem of Kuratowski and Ryll-Nardzweski [7]. It does not hold in general for $\alpha = 0$. But we do have in that case the following theorem on convex-valued multifunctions which in turn contains two selection theorems of Michael [9].

2.3. THEOREM. Suppose Y is a metrizable locally convex space, $\phi \in \mathcal{M}_c^*(X,Y)$ and that one of the following two conditions holds:

(i) X is normal, Y is separable and ϕ is complete-valued,

(ii) X is collectionwise normal and ϕ is compact-valued.

Then ϕ is LSC iff it is the limit of a nondecreasing sequence of elementary multifunctions $\{\phi_n\}$ whose elements are continuous.

For the multifunctions in $\mathbf{UB}_\alpha$ we have similarly the following characterizations which hold for $\alpha > 0$ and $\alpha = 0$.

2.4. THEOREM. Suppose Y is a locally compact separable metric space, $\phi \in \mathcal{M}_k(X,Y)$ and that $\alpha > 0$. Then $\phi \in \mathbf{UB}_\alpha$ iff it is the limit of a nonincreasing sequence of compact-valued multifunctions $\{\phi_n\}$ in $\mathbf{B}_\alpha$.

2.5. THEOREM. Suppose X is normal, Y is a separable normed vector space, $\phi \in \mathcal{M}_{k,c}^*(X,Y)$ and that there exists a continuous multifunction ψ in $\mathcal{M}_{k,c}(X,Y)$ such that $\phi \subset \psi$. Then ϕ is USC iff it is the limit of a nonincreasing sequence of continuous multifunctions $\{\phi_n\}$ in $\mathcal{M}_{k,c}(X,Y)$.

3. <u>Representation of multifunctions in $\mathbf{B}_\alpha$ as limits of elementary multifuctions in lower Baire classes</u>. Let us state first a general result on the Baire class of limits of sequences of multifunctions.

3.1. THEOREM. Suppose Y is perfectly normal and let $\{\phi_n\}$ be a sequence of multifunctions in $\mathcal{M}(X,Y)$ which converges to ϕ.

(a) If $\phi_n \in \mathbf{UB}_\alpha$ for each n, then $\phi \in \mathbf{LB}_{\alpha+1}$.

(b) If ϕ is compact-valued and $\phi_n \in \mathbf{LB}_\alpha$ for each n, then $\phi \in \mathbf{UB}_{\alpha+1}$.

(c) If ϕ is compact-valued, Y is second countable and $\phi_n \in \mathbf{B}_\alpha$ for each n, then $\phi \in \mathbf{B}_{\alpha+1}$.

The following theorem extends a well-known theorem on functions [5, p. 390] to multifunctions and it is easy to see that the result on functions is contained in this theorem.

3.2. THEOREM. Suppose Y is separable and metrizable, $\phi \in \mathcal{M}_k^*(X,Y)$ and that $\alpha > 1$. If $\phi \in \mathbf{B}_\alpha$, then ϕ is the limit of a sequence of elementary multifunctions $\{\phi_n\}$ whose elements are in Baire classes lower than α.

Moreover, if $\alpha = \lambda+1$ where λ is a limit ordinal, then the elements of each ϕ_n can be chosen to be in Baire classes lower than λ.

In the case when $\alpha = 1$ we have, on the other hand,

3.3. THEOREM. Suppose $\phi \in \mathcal{M}_k^*(X,Y)$, where X is normal and Y is a separable metrizable asbolute retract (for metrizable spaces). Then $\phi \in \mathbf{B}_1$ iff it is the limit of a sequence of elementary multifunctions $\{\phi_n\}$ whose elements are continuous.

The above theorem contains a similar result on functions which generalizes the existing results in that direction (see e.g. [5, p. 391] and Banach [2]).

Next, we obtain with the help of the above theorems an extension of the classical Lebesgue-Hausdorff theorem [5, p. 393] to multifunctions. The analytic classes of multifunctions are defined as usual using transfinite induction. The analytic class 0 consists of all continuous multifunctions and, for each countable ordinal $\alpha > 0$, the analytic class α is defined to be the class of all pointwise limits of sequences of multifunctions of analytic classes lower than α.

3.4. THEOREM. Suppose $\phi \in M_k^*(X,Y)$, where X is normal and Y is a separable metrizable absolute retract. Then ϕ is in analytic class α iff it is in $\mathbf{B}_\alpha$ or $\mathbf{B}_{\alpha+1}$ according as α is finite or infinite.

The same holds for a function $f: X \to Y$ in terms of its analytic class as a function.

The last part of the above theorem generalizes all the existing versions of the Lebesgue-Hausdorff theorem due to Lebesgue [8], Hausdorff [4], Kuratowski [5, p. 393] and Banach [2].

4. <u>Interposition theorems</u>. We present here two interposition theorems on multifunctions which are similar to the interposition theorem of Hahn on real-valued functions (see [4, p. 281]).

4.1. THEOREM. Suppose $\phi, \psi \in M(X,Y)$, where Y is a separable metric space, ϕ is compact-valued and ψ is complete-valued, and let $\alpha > 0$. If $\phi \in \mathbf{UB}_\alpha$, $\psi \in \mathbf{LB}_\alpha$ and $\phi \subset \psi$, then there exists a multifunction $\theta \in \mathbf{B}_\alpha$ such that $\phi \subset \theta \subset \psi$.

The above theorem does not hold in general for $\alpha = 0$. But we do have in that case the following theorem on convex-valued multifunctions. The weak continuity refers here to the continuity relative to the Vietoris topology generated by the weak topology of Y.

4.2. THEOREM. Suppose X is normal, Y is a separable reflexive normed vector space, $\phi, \psi \in M_c(X,Y)$ and that one of the following conditions holds:

(i) ϕ is compact-valued and ψ is nonempty-valued,

(ii) ϕ has nonempty weakly compact values and the dual space Y^* is separable.

If $\phi \in USC$, ψ is LSC and $\phi \subset \psi$, then there exists a weakly continuous multifunction $\theta \in M_c^*(X,Y)$ with weakly compact values such that $\phi \subset \theta \subset \psi$.

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