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QUALITATIVE ASPECTS OF DIFFERENTIATION

The talk with the above title, which was presented at the Sixth Summer Symposium on Real Analysis at the University of Waterloo, was largely based on [5], a joint work with Lee Larson. This article is a summary of that presentation, supplemented with some subsequent observations. The notions of qualitative limits, qualitative continuity, and qualitative derivatives were introduced by S. Marcus [9 - 11]. Loosely speaking, qualitative differentiation may be thought of as an attempt at a category analogue of approximate differentiation, where the set neglected near a point in the computation of difference quotients if of first category at the point in the former setting instead of density zero at the point as in the latter.

This qualitative limiting process of Marcus is not a direct analogue of the approximate process in that it is not sufficiently delicate to capture the analogue of a point of density. (A potentially more promising approach has recently been advanced by W. Wilczyński [15].) There are, nonetheless, numerous parallels between approximate and qualitative notions. For example, corresponding to the well-known fact that a function $f : R \to R$ is Lebesgue measurable if and only if it is approximately continuous except at a measure zero set of points, we have that a function $f : R \to R$ has the property of Baire ($f \in \beta$) if and only if it is qualitatively continuous except at a first category set of points [5]; similarly, corresponding to the result of R. J. O'Malley [12] that the set of points at which $f : R \to R$ attains a strict approximate maximum is of measure zero, we have that the set of points at which f attains a strict qualitative maximum is of first category [4]. In this current presentation we wish to concentrate on qualitative differentiation.

Corresponding to the upper right Dini derivate of f at x, $D^+f(x)$, we let $Q^+f(x)$ denote the upper right qualitative derivate of f at x; i.e.,

$$Q^+f(x) = \inf\{y : \{t : f(t) - f(x) > y(t - x)\} \text{ is first category in some right neighborhood of } x\}.$$

Then $Q_+f(x)$, $Q^-f(x)$, and $Q_-f(x)$ are defined analogously and if all four are equal, the common value is called the qualitative derivative of f at x and is denoted by $f_q'(x)$. If we use symmetric difference quotients, we may similarly define upper and lower symmetric qualitative derivates, $Q^sf(x)$ and $Q_sf(x)$, and in the situation where these two are equal, denote their common value by $f_q^s(x)$, the qualitative symmetric derivative of f and x. It is convenient to let $C_q(f)$ denote the set of points at which f is qualitatively continuous.

Several authors have observed that if a function has a finite qualitative derivative everywhere on R, then it actually has an ordinary derivative everywhere on R [1,2,8]. Using the approach taken by J. L. Leonard [8], it is easy to see that if f has a finite right qualitative derivative at each point of R, then it has an ordinary right derivative everywhere. Leonard also observed that finiteness is necessary in this result. In [5] it is shown that finiteness is not needed in the bilateral case.

Theorem 1. *If $f : R \to R$ has a qualitative derivative everywhere, then it has an ordinary derivative everywhere.*

Examples of symmetry results that can be obtained for qualitative derivates are the following:

Theorem 2. *If $f : R \to R$ is arbitrary and*

$$A = \{x : Q^{-}f(x) = Q^{+}f(x) \text{ and } Q_{-}f(x) = Q_{+}f(x)\}$$

$$B = \{x : -\infty = Q_{-}f(x) \le Q_{+}f(x) \le Q^{-}f(x) \le Q^{+}f(x) = \infty\}$$

$$C = \{x : -\infty = Q_{+}f(x) \le Q_{-}f(x) \le Q^{+}f(x) \le Q^{-}f(x) = \infty\}$$

then (i) $R\setminus(A \cup B \cup C)$ *is first category,*

(ii) $C_q(f)\setminus(A \cup B \cup C)$ *is σ-porous,*

(iii) *if $f \in \beta$, $R\setminus A$ is first category,*

and (iv) *if f is monotone on a residual set S, then $S\setminus A$ is σ-porous.*

From Theorem 2, part (ii), it readily follows that the set of points at which a function has a finite one-sided qualitative derivative but does not have a qualitative derivative is σ-porous, paralleling a result of L. Zajíček [16] concerning approximate differentiation. An alternate approach for obtaining this and related qualitative results is explored in [3], where a result involving qualitative angular cluster sets is proved and then coupled with the Jarnik-Blumberg method to show that if A is as in Theorem 2 and

$$D = \{x : \max\{|Q_+f(x)|, |Q^+f(x)|\} = \max\{|Q_-f(x)|, |Q^-f(x)|\} = \infty\},$$

then $R\setminus(A \cup D)$ is σ-porous. This latter result again parallels the approximate case [16].

During the Symposium at the University of Waterloo, Professor James Foran called my attention to a recent paper of D. Preiss and L. Zajíček [13], in which a significant improvement on the result alluded to above concerning the symmetry of approximate derivates is made. I wish to take this opportunity to show how the qualitative version of their result may easily be obtained from the results noted to this point by proving the following theorem.

Theorem 3. *If* $f : R \to R$ *is arbitrary, if* A *is as in Theorem 2, and if*

$$E = \{x : Q_-f(x) = -\infty \ \text{ and } \ Q^+f(x) = +\infty\}$$

and

$$F = \{x : Q_+f(x) = -\infty \ \text{ and } \ Q^-f(x) = +\infty\},$$

then $R\setminus(A \cup E \cup F)$ *is* σ*-porous.*

The proof will rest on two simple lemmas.

Lemma 1. *Let* $f : R \to R$ *be arbitrary and let* $S = \{x : \max\{Q^+f(x), Q^-f(x)\} < +\infty\}$. *Then* $S \setminus C_q(f)$ *is countable.*

Proof. It is an easy matter to see that the set

$$T = \{x : q\text{-}\overline{\lim_{t \to x^-}}\, f(t) \neq q\text{-}\overline{\lim_{t \to x^+}}\, f(t) \text{ or } q\text{-}\underline{\lim_{t \to x^-}}\, f(t) \neq q\text{-}\underline{\lim_{t \to x^+}}\, f(t)\}$$

is countable, where $q\text{-}\overline{\lim_{t \to x^-}}\, f(t)$ denotes the qualitative limit superior of f at x from the left and so forth. (In [4] a slightly more general statement was proved, indicating that the set of points at which the left and right qualitative cluster sets are not identical is countable.)

Now, let $x \in S \setminus T$. Since $Q^+f(x) < +\infty$, it follows that

$$q\text{-}\overline{\lim_{t \to x^+}}\, f(t) \leq f(x),$$

and, similarly, since $Q^-f(x) < +\infty$,

$$q\text{-}\underline{\lim_{t \to x^-}}\, f(t) \geq f(x).$$

From these two inequalities and the assumption that $x \notin T$, it follows that $x \in C_q(f)$, completing the proof of this lemma.

Lemma 2. *Let* $f : R \to R$ *be arbitrary and let* A *and* S *be as defined in Theorem 2 and Lemma 1, respectively. Then* $S \setminus A$ *is* σ*-porous.*

Proof. This follows immediately from Lemma 1, Theorem 2 (part (ii)), and the observation that

$$S \cap B = S \cap C = \emptyset.$$

Turning now to the proof of Theorem 3, recall that we already know

$$R\setminus(A \cup D) \text{ is } \sigma\text{-porous}$$

and, hence, that

(1) $$[R\setminus(A \cup E \cup F)]\setminus D \text{ is } \sigma\text{-porous},$$

where A,D,E, and F are as described previously.

Next, let $x \in [R\setminus(A \cup E \cup F)] \cap D$. Then, if we let S be as defined in Lemma 1 and S' be the analogous set for the function -f, it is easy to see that $x \in S \cup S'$. It then follows from Lemma 2 that

(2) $$[R\setminus(A \cup E \cup F)] \cap D \text{ is } \sigma\text{-porous}.$$

The theorem follows from (1) and (2).

It should be noted that the arguments [16 and 13] needed to establish the approximate version of Theorem 3 are significantly less trivial than those needed in the present qualitative setting.

Concerning qualitative symmetric derivates, we obtain the following [5]:

Theorem 4. If $f : R \to R$ is in β and if V denotes the set of points x at which both

$$Q_s f(x) = \min\{Q_- f(x), Q_+ f(x)\} \text{ and } Q^s f(x) = \max\{Q^- f(x), Q^+ f(x)\},$$

then $C_q(f)\setminus V$ is σ-porous.

It readily follows that if $f \in \beta$ and f has a qualitative symmetric derivative everywhere, then f_q' exists at a residual set of points. It is also shown [5] that such a qualitative symmetric derivative is in Baire class one and that if this derivative is non-negative on R, then f must be nondecreasing on its set of points of qualitative continuity. Techniques employed for these latter results and related theorems are based on those used by Larson in [6 and 7] and Thomson in [14].

References

1. A. M. Bruckner, Differentiation of real functions, Lecture Notes in Mathematics, No. 659, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

2. A. M. Bruckner, R. J. O'Malley, and B. S. Thomson, Path derivatives: a unified view of certain generalized derivatives, Trans. Amer. Math. Soc. (to appear).

3. M. J. Evans, Qualitative derivates and derivatives, Rev. Roum. Math. Pures et Appl. (to appear).

4. __________, Qualitative maxima of a function, Rev. Roum. Math. Pures et Appl. (to appear).

5. M. J. Evans and L. Larson, Qualitative differentiation, Trans. Amer. Math. Soc. (to appear).

6. L. Larson, The Baire class of approximate symmetric derivates, Proc. Amer. Math. Soc. 87(1983), 125-130.

7. _________, The symmetric derivative, Trans. Amer. Math. Soc. 277(1983), 589-599.

8. J. L. Leonard, Some conditions implying the monotonicity of a real function, Rev. Roum. Math. Pures et Appl. 12(1972), 757-780.

9. S. Marcus, Sur la limite approximative qualitative, Com. Acad. R. P. Romine 3(1953), 9-12. (Roumanian; French summary.)

10. _________, Sur la continuité approximative qualitative, Com. Acad. R. P. Romine 3(1953), 117-120. (Roumanian; French summary.)

11. _________, Sur la dérivée approximative qualitative, Com. Acad. R. P. Romine 3(1953), 361-364. (Roumanian; French summary.)

12. R. J. O'Malley, Strict essential minima, Proc. Amer. Math. Soc. 33(1972), 501-504.

13. D. Preiss and L. Zajíček, On the symmetry of approximate Dini derivates of arbitrary functions, Comment. Math. Univ. Carolinae 23(1982), 691-697.

14. B. S. Thomson, On full covering properties, Real Anal. Exchange 6(1980), 77-93.

15. W. Wilczyński, A generalization of the density topology, Real Anal. Exchange 8(1982), 16-20.

16. L. Zajíček, On approximate Dini derivates and one sided approximate derivatives of arbitrary functions, Comment. Math. Univ. Carolinae 22(1981), 549-560.