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Fourier Integral Inequalities and Applications

1. Let $\mathcal{F}$ denote the Fourier transform defined by

$$(\mathcal{F}f)(x) \equiv \int_{-\infty}^{\infty} e^{-2\pi ixy} f(y)dy, \qquad x \in \mathbb{R},$$

where f belongs to a suitable function space for which the operator exists. If L^p, $p \geq 1$, denote the usual Lebesgue spaces with norm $\|\cdot\|_p$, then it is obviou that $\mathcal{F}: L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded and Plancherel's theorem shows that $\mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded. It is then easy to deduce from the Riesz-Thorin interpolation theorem that the Hansdorff-Young inequality $\|\mathcal{F}f\|_{p'} \leq \|f\|_p$, $1 < p < 2$, holds, where now and in the sequel p' denotes the conjugate index of p and the two are related by $p + p' = pp$, with $p' = 1$ if $p = \infty$.

There are many weighted extensions of this inequality of the form

$$(1.1) \qquad \left(\int_{\infty}^{\infty} u(x)\,|(\mathcal{F}f)(x)|^q dx\right)^{1/q} \leq C \left(\int_{\infty}^{\infty} v(x)\,|f(x)|^p dx\right)^{1/p},$$

for some p,q, $1 < p, q \leq \infty$ and u and v are non-negative weight functions. For example, if $u \equiv 1$ and $v(x) = |x|^{q-2}$, and $p = q > 2$, or $v \equiv 1$, $u(x) = |x|^{p-2}$, $1 < p < 2$, $q = p$, in (1.1) one obtains Titchmarsh's extensions of the Hansdorff-Young inequality [21]. On the other hand, if $u = v \equiv 1$ and $p > 2$ then there exists a function for which the right side of (1.1) is finite, but $\mathcal{F}f$ cannot exist as a function [18; p. 34 (4,13)]. This phenomena is reflected by

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by other integral operators and it raises the following question:

Given a linear operator T defined on some suitable function space, what are necessary and sufficient conditions on non-negative weight functions u and v, such that

$$(1.2) \qquad \left(\int_\Omega u(x)\,|(Tf)(x)|^q dx\right)^{1/q} \leq C\left(\int_\Omega v(x)\,|f(x)|^p dx\right)^{1/p},$$

where $1 \leq p,\ q \leq \infty$ and $\Omega \subseteq \mathbb{R}^n$?

In this note we consider this question when $n = 1$ and T is the Hardy operator (and its dual), the Laplace transform and the Fourier transform.

In the next section we discuss the solution of this problem for the Hardy operator and show that the case $q > p$ follows via a simple lemma from the $q = p$ case. The principal but partial results for the Laplace and Fourier transform are also given there. The final section contains some applications. Specifically an inequality related to Hilbert's inequality is given and a generalization of the Heisenberg-Weyl inequality is obtained.

Throughout C denotes a constant independent of the function in question, but may be different at different occurrences. Sometimes we also write for $(-\infty,\infty)$, $\mathbb{R}$ and $\mathbb{R}^+ = (0,\infty)$.

2. Let P denote the Hardy operator and $\overline{P}$ its dual, that is,

$$(Pf)(x) = \int_0^x f(t)dt, \qquad (\overline{P}f)(x) = \int_x^\infty f(t)dt, \qquad x > 0,$$

where, for the purpose of the discussion we take without loss of generality $f \geq 0$. It is now well known [3][12][19][20] that (1.2) holds with $T = P$ or $T = \overline{P}$ with $p = q \geq 1$ and $\Omega = \mathbb{R}^+$, if and only if for all $s > 0$

$$\left(\int_0^\infty e^{-sxq}u(x)dx\right)^{1/q} \le \lim_{h\to 0}\left(\int_0^\infty u(x)\left[\frac{1}{2h}\int_{s-h}^{s+h} e^{-xy}dy\right]^q dx\right)^{1/q} \le C\lim_{h\to 0}\frac{1}{2h}\int_{s-h}^{s+h} v(x)dx = Cv(s)$$

provided s is a Lebesgue point of v. This implies the result.

Although a complete characterization of weights for the Laplace transform seems to be not known, it has recently been shown by E. Sawyer that under very mild conditions on the weight u such a characterization exists.

The question of characterizing weights u,v when $T = \mathcal{F}$ the Fourier transform was raised in [13]. Recently several authors independently obtained significant results in this direction [14],[15],[9],[11]. In order to state the result we introduce some notation.

Given a Lebesgue measurable function, then the equimeasurable decreasing rearrangement of f is defined by $f^*(t) = \inf\{y > 0: m(\{x \in \mathbb{R} : |f(x)| > y\}) \le t\}$, $t > 0$ where m denotes Lebesgue measure. The symmetrically decreasing rearrangement of f is then defined by $f^{\circledast}(t) = f^*(2t)$ if $t > 0$ and extended as an even function on $\mathbb{R}$. We further write $(u,v) \in F(p,q)$ $1 \le p \le q \le \infty$ if

$$\sup_{s>0}\left(\int_0^{1/(2s)} u^{\circledast}(t)^q dt\right)^{1/q}\left(\int_0^{s/z} (1/v)^{\circledast}(t)^{p'}dt\right)^{1/p'} < \infty,$$

(with the usual modification if $p = 1$ and/or $q = \infty$.)

Theorem 2.3. ([9],[11],[14],[15],[4]) If $(u,v) \in F(p,q)$, $1 \le p \le q \le \infty$, $p <$ then

$$\left(\int_{-\infty}^\infty |u(x)(\mathcal{F}f)(x)|^q dx\right)^{1/q} \le C\left(\int_{-\infty}^\infty |v(x)f(x)|^p dx\right)^{1/p}. \tag{2.6}$$

(2.4) $$\left(\int_0^\infty u(x)(\mathcal{L}f)(x)^q dx \right)^{1/q} \le C \left(\int_0^\infty v(x)f(x)^p dx \right)^{1/p} .$$

If in addition it is assumed that u is decreasing and v increasing, then (2.3) is necessary and sufficient for (2.4).

We note again that this result much as in the case of the Hardy operator may be obtained from the case $q = p$ via Lemma 2.1 (i). For the case $p = 1$ the following result is sharper.

<u>Theorem 2.2.</u> Let $1 \le q \le \infty$, then (2.4) with $p = 1$ holds if and only if

(2.5) $$\sup_{s>0} [(\mathcal{L}u)(sq)]^{1/q}/v(s) < \infty ,$$

with the obvious modification if $q = \infty$.

<u>Proof.</u> The first part is immediate since

$$\left(\int_0^\infty u(x) \left(\int_0^\infty e^{-xy} f(y)dy \right)^q dx \right)^{1/q} \le \int_0^\infty f(y) \left(\int_0^\infty e^{-xyq} u(x)dx \right)^{1/q} dy$$

$$\le C \int_0^\infty v(y)f(y)dy$$

by Minkowski's integral inequality and (2.5).

Conversely, let $f(y) = \chi_{(s-h,s+h)}(y)/(2h)$, $h > 0$ and $s - h > 0$, where χ denotes the characteristic function. Then (2.4) with $p = 1$ becomes

$$\left(\int_0^\infty u(x) \left[\frac{1}{2h} \int_{s-h}^{s+h} e^{-xy} dy \right]^q dx \right)^{1/q} \le C \left(\frac{1}{2h} \int_{s-h}^{s+h} v(x)dx \right) .$$

As $h \to 0$, Fatou's lemma implies that replaced by

$$u(x)\left(\int_x^\infty u(t)dt\right)^{p/q-1}.$$

But then an integrating

$$\left(\int_s^\infty u(x)\left(\int_x^\infty u(t)dt\right)^{p/q-1}dx\right)^{1/p}\left(\int_0^s v(x)^{-p'/p}dx\right)^{1/p'}$$

$$= \quad (q/p)^{1/p}\left(\int_s^\infty u(t)dt\right)^{1/q}\left(\int_0^s v(x)^{-p'/p}dx\right)^{1/p'} \leq C < \infty$$

The boundedness of $\overline{P}: L_v^p(\mathbb{R}^+) \to L_u^q(\mathbb{R}^+)$, $p < q$ follows also from the case $q = p$ in the same way, only now one uses (i) of Lemma 2.1 .

These extensions using a different proof may be found in [5] and [2]. Characterizations of weights for the Hardy operator in the case where $0 < q < p$ were recently obtained by E. Sawyer [17], but there the situation is not as simple.

Next, we consider the Laplace transform defined by

$$(\mathcal{L}f)(x) = \int_0^\infty e^{-xy}f(y)dy, \qquad x > 0 ,$$

where again for the purposes of our discussion we take without loss of generality $f \geq 0$. The following partial result is known:

<u>Theorem 2.1.</u> [1; Theorem 2.4] Suppose $1 \leq p \leq q \leq \infty$,

$$\sup_{s>0}\left(\int_0^{1/s} u(x)dx\right)^{1/q}\left(\int_0^s v(x)^{-p'/p}dx\right)^{1/p'} < \infty \tag{2.3}$$

and for some β, $0 \leq \beta \leq 1$

$$\sup_{s>0}\left(\int_{1/s}^\infty e^{-\beta sqx}u(x)dx\right)^{1/q}\left(\int_s^\infty e^{-(1-\beta)p'x/s}v(x)^{-p'/p}dx\right)^{1/p'} < \infty$$

then

$$I^p \geq \int_0^y u(x)\left(\int_0^x u(t)dt\right)^{p/q-1} h(x)^p dx \geq h(y)^p (q/p)\left(\int_0^y u(t)dt\right)^{p/q},$$

so that

$$h(y) \leq (p/q)^{1/p}\, I\left(\int_0^y u(t)dt\right)^{-1/q}.$$

Therefore

$$\left(\int_0^\infty u(x)h(x)^q dx\right)^{1/q} = \left(\int_0^\infty u(x)h(x)^p h(x)^{q-p} dx\right)^{1/q}$$

$$\leq (p/q)^{(q-p)/(pq)}\, I^{(q-p)/q}\left(\int_0^\infty u(x)h(x)^p\left(\int_0^x u(t)dt\right)^{p/q-1} dx\right)^{1/q}$$

$$= (p/q)^{1/p-1/q}\, I$$

which proves this part of the Lemma.

For the second part one denotes the right integral by J^p and observes that for $y > 0$

$$J^p \geq \int_y^\infty u(x)\left(\int_x^\infty u(t)dt\right)^{p/q-1} h(x)^p dx = (p/q)h(y)^p\left(\int_y^\infty u(t)dt\right)^{p/q}.$$

Then one proceeds as before.

To show now that $P: L_v^p(\mathbb{R}^+) \to L_u^q(\mathbb{R}^+)$, $1 \leq p \leq q \leq \infty$ is bounded one uses Lemma 2.1 (ii) with $h(x) = (Pf)(x)$. Then

$$\left(\int_0^\infty u(x)(Pf)(x)^q dx\right)^{1/q} \leq (p/q)^{1/p-1/q}\left(\int_0^\infty u(x)\left(\int_x^\infty u(t)dt\right)^{p/q-1}(Pf)(x)^p dx\right)^{1/p}$$

$$\leq C\left(\int_0^\infty v(x)f(x)^p dx\right)^{1/p}$$

where the last inequality follows from the case $q = p$, provided (2.1) holds with u

$$(2.1) \qquad \left(\int_s^\infty u(x)dx\right)^{1/p} \left(\int_0^s v(x)^{-p'/p}\,dx\right)^{1/p'} \le C < \infty,$$

respectively

$$(2.2) \qquad \left(\int_0^s u(x)dx\right)^{1/p} \left(\int_s^\infty v(x)^{-p'/p}\,dx\right)^{1/p'} \le C < \infty ,$$

with the obvious modification in the second integral of (2.1) and (2.2) if $p = 1$.

Observe that if one takes $u(x) = x^{-r-1}$, $v(x) = x^{-r-1+p}$, $r > 0$ or in case $T = \overline{P}$, $u(x) = x^{r-1}$, $v(x) = x^{r-1+p}$, $r > 0$ one obtains the celebrated inequalities of Hardy [7; 245-246]. The importance of these inequalities and their generalizations lies in the fact that the Marcinkiewicz interpolation theorem and its weighted generalizations are consequences of them. (See e.g. [10],[8],[16],[9].)

The following elementary Lemma shows that the sufficiency part of the result for the Hardy operator and its dual, as well as those for the Laplace transform, extends easily to the case $q > p$.

Lemma 2.1. Let h and u be non-negative functions on $\mathbb{R}^+$ and $0 < p < q < \infty$.

(i) If h is decreasing, then

$$\left(\int_0^\infty u(x)h(x)^q dx\right)^{1/q} \le (p/q)^{1/p-1/q} \left(\int_0^\infty u(x)\left(\int_0^x u(t)dt\right)^{p/q-1} h(x)^p dx\right)^{1/p}.$$

(ii) If h is increasing, then

$$\left(\int_0^\infty u(x)h(x)^q dx\right)^{1/q} \le (p/q)^{1/p-1/q}\left(\int_0^\infty u(x)\left(\int_x^\infty u(t)dt\right)^{p/q-1} h(x)^p dx\right)^{1/p}.$$

Proof. (i) We assume without loss of generality that the integral on the right is finite. We denote it by I^p. Then for any $y > 0$ one obtains an integrating

Moreover, if u and v are even functions $u(t)$ decreasing for $t > 0$ and $v(t)$ increasing for $t > 0$, that is $u = u^{\circledast}$ and $1/v = (1/v)^{\circledast}$, then $(u,v) \in F(p,q)$ is necessary and sufficient for (2.6).

3. As a consequence of Theorem 2.2 we have

Corollary 3.1

$$\int_0^{\infty}\int_0^{\infty}(\mathscr{L}u)(x+y)f(x)f(y)\,dxdy \le C\left(\int_0^{\infty}v(x)f(x)dx\right)^2$$

if and only if (2.5) holds with $q = 2$.

Proof. The left side of (3.1) can be written as

$$\int_0^{\infty}\int_0^{\infty}f(x)f(y)\left(\int_0^{\infty}e^{-(x+y)t}u(t)dt\right)dx\,dy = \int_0^{\infty}u(t)(\mathscr{L}f)(t)^2dt \le C\left(\int_0^{\infty}v(x)f(x)dx\right)^2$$

Here the last inequality holds if and only if (2.5) is satisfied. Specifically taking $u(x) = x^{\lambda-1}$, $v(x) = x^{-\lambda/2}$, $\lambda > 0$ then (2.5) is satisfied with $q = 2$ and Corollary 3.1 yields

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)f(y)}{(x+y)^{\lambda}}\,dxdy \le C\left(\int_0^{\infty}x^{-\lambda/2}f(x)dx\right)^2$$

This inequality should be compared with Hilberts inequality [7; p.226].

If S is the Stieltjes transform defined by

$$(Sf)(x) = \int_0^{\infty}\frac{f(y)}{x+y}\,dy\ , \qquad f \ge 0,\ x > 0\ ,$$

then one obtains in the same way

Corollary 3.2

$$\int_0^\infty u(x)(Sf)(x)dx \le C \int_0^\infty v(x)f(x)dx$$

if and only if $\sup_{s>0}(Su)(s)/v(s) < \infty$.

In [4], Theorem 2.3 was applied to obtain new representation theorems for the Laplace transform via weighted Hardy spaces. Here we use it to give a weighted extension of the Heisenberg-Weyl inequality.

Fundamental to Heisenberg's uncertainty principle is the inequality

$$\int_{-\infty}^{\infty} |f(x)|^2 dx \le 4\pi \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \right)^{1/2} \left(\int_{-\infty}^{\infty} y^2 |(\mathcal{F}f)(y)|^2 dy \right)^{1/2}$$

where $f \in L^2(\mathbb{R})$. As a corollary to Theorem 2.3 we prove

Corollary 3.3. If $(1/u,v) \in F(p,q)$, $1 \le p, q < \infty$ and $f \in L^2(\mathbb{R})$, not identically zero, then $\|f\|_2^2 \le C\|xf\|_{u,q'}\|y(\mathcal{F}f)\|_{v,p'}$ where $\|g\|_{w,r} = \|wg\|_r$.

Proof. It sufficies to prove this result for $f \in \mathcal{S}$, the Schwartz space. Let g' denote the derivative of g, then for $p \le q$, by Holders inequality and Theorem 2.3

$$\|f\|_2^2 = \int_{-\infty}^{\infty} x(|f(x)|^2)'dx \le 2 \int_{-\infty}^{\infty} |xu(x)\overline{f(x)}||f'(x)/u(x)|dx$$

$$\le 2\|xf\|_{u,q'}\|f'\|_{1/u,q} \le C\|xf\|_{u,q'}\|(\mathcal{F}f')\|_{v,p} = C\|xf\|_{u,q'}\|y(\mathcal{F}f)\|_{v,p}.$$

If $q < p$, then $p' < q'$ and since $(1/u,v) \in F(p,q)$ implies $(1/v,u) \in F(q',p')$ Plancherel's theorem and the same argument yields

$$\|f\|_2^2 = \|\mathcal{F}f\|_2^2 \le 2\int_{-\infty}^{\infty} |yv(y)(\overline{\mathcal{F}f})(y)||(\mathcal{F}f)'(y)/v(y)|dy$$

$$\le 2\|y(\mathcal{F}f)\|_{v,p}\|(\mathcal{F}f)'\|_{1/v,p'} = 2\|y(\mathcal{F}f)\|_{v,p}\|\mathcal{F}(xf)\|_{1/v,p'}$$

$$\le C\|y(\mathcal{F}f)\|_{v,p}\|xf\|_{u,q'} ,$$

which is the result.

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