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Balanced Selections

A selection s can be thought of as an interval function defined on the class of all nondegenerate closed subintervals [a,b] of R with the property that for each [a,b], $a < s[a,b] < b$. The author has used selections to consider a form of generalized derivative, [1]. A function $f : R \to R$ is said to have another function $g : R \to R$ as selective derivative, with respect to a selection s, if for each x

$$\lim_{h\to 0} \frac{f(s[x,x+h]) - f(x)}{s[x,x+h] - x} = g(x).$$

(If $h < 0$, then [x,x+h] denotes the interval [x+h,x].)

One contrast between a selective derivative and an ordinary derivative is the fact that some selective derivatives are strictly Baire class 2. In this paper, we investigate how this happens. More precisely, we introduce a concept, which we call balance, into the study of selections. It will be shown that for a selective derivative to be strictly Baire class 2, the selection must be unbalanced. Intuitively the idea of balance is that the point s[a,b] should not be too close to either a or b. For example, if s[a,b] was always in the middle third of the interval [a,b] the selection will be considered balanced.

We will need the following convention. Let [a,b] be a fixed interval and α a fixed number $0 < \alpha < 1$. By the α interval of [a,b] we mean the interval centered in [a,b] of length $\alpha(b-a)$.

Definition: A section s is said to be balanced if there are two functions $\alpha(x)$ and $\delta(x)$ such that $0 < \alpha(x)$, $0 < \delta(x)$, and if $I = [a,b]$ is any interval having x as one end point and $b-a < \delta(x)$, then $s[a,b]$ is in the $\alpha(x)$ interval of $[a,b]$.

For the example above we have an uniformly balanced selection in which we could take $\alpha(x) = 1/3$ and $\delta(x) = 1$ for all x. As a note, we point out that in [1] an example of a strictly Baire class 2 selective derivative is given. A perusal of that example would show how the selection unbalances. However the presentation here would not be warranted.

Theorem: If $f : R \to R$ has a selective derivative $g(x)$ with respect to a selection s which is balanced relative to functions $\alpha(x)$, $\delta(x)$, then g is Baire class 1.

<u>Proof.</u> Let P be any perfect set and let $\varepsilon > 0$ be fixed. It will suffice to find a portion of P and numbers u', v' such that $u' < g(x) < v'$ for all x in the portion and $v' - u' < \varepsilon$. By a portion of P we mean a set of the form $(c,d)\cap P \neq \emptyset$.

Define for each n, $A_n = \{x : \alpha(x) < 1 - 1/n \text{ and } \delta(x) > 1/n\}\cap P$. Using the Baire category theorem we can find an integer N and a portion P_1 of P with A_N residual in P_1. We may assume that the diameter of P_1 is less than $1/N$.

For this N and the ε above we consider all pairs of rationals u,v with $0 < (v-u)(2N-1) < \varepsilon/3$. For each such pair and integer k define

$$B_{uvk} = \{x: u < \frac{f(s[x,x+h] - f(x)}{s[x,x+h] - x} < v \text{ when } |h| < 1/k\}\cap P_1.$$

Then the union of the countable collection of sets B_{uvk} will be P_1. Therefore we are guaranteed the existence of a new portion P_2 of P_1, a pair u,v and an integer k such that B_{uvk} is residual in P_2. We may assume the diameter of P_2 is less than 1/k. This in turn, by a standard argument as in lemma 3 [1], implies that for all x,y in P_2 we have:

$$u \leq \frac{f(y) - f(x)}{y - x} \leq v.$$

We claim that actually $u' = u - \varepsilon/3 \leq g(x) \leq v + \varepsilon/3 = v'$ which would prove the theorem as $v - u < \varepsilon/3$.

Let x belong to P_2. The intersection $A_N \cap B_{uvk} \cap P_2$ is residual in P_2. This allows us to find a sequence of points x, from this intersection, which converges to x.

We may assume that

a) $x = 0$ and $f(0) = 0$.

b) $x_j > x$ for all j, and

c) $u > 0$.

It should be noted that the following items are true for all j:

i) $x_j < \min(1/k, 1/N)$,

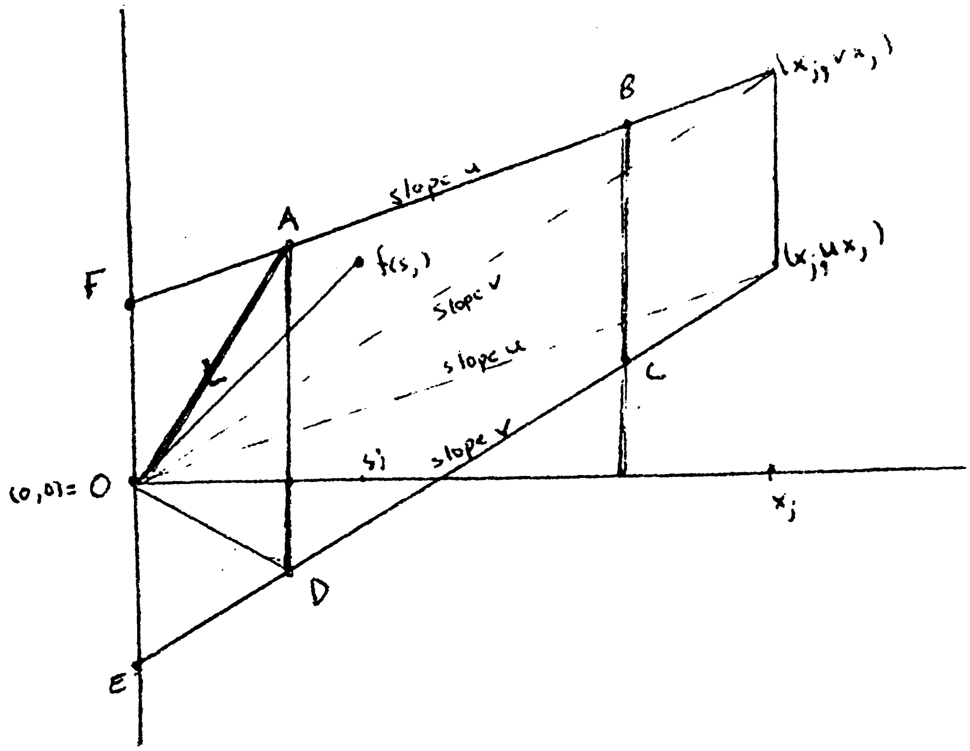
ii) $ux_j \leq f(x_j) \leq vx_j$,

iii) $s[0,x] = s_j$ is in the (1-1/N) interval of $[0,x_j]$, that is

iii*) $\frac{1}{2N} x_j < s_j < (1-1/2N)x_j$

iv) $f(x_j) - v(x_j-s_j) \leq f(s_j) \leq f(x_j) - u(x_j-s_j)$

Using these assumptions and items the following picture holds:



The point $(s_j, f(s_j))$ is in the quadrilaterial A B C D. The origin (0,0) will be inside quadrilateral F B C E. Thus, the line segment L from origin to $(s_j, f(s_j))$ must intersect line segment AD. More importantly, the slope of L will be bounded below and above respectively by the slopes of line segments OD and OA. We have:

$$\text{slope OD} = \frac{u(x_j) - v(x_j - \frac{1}{2N} x_j)}{\frac{1}{2N} x_j} = u_0 = u - (2N-1)(v-u).$$

and

$$\text{slope OA} = \frac{v(x_j) - u(x_j - \frac{1}{2N} \lambda_j)}{\frac{1}{2N} x_j} = v_0 = v + (2N-1)(v-u).$$

so $\quad u_0 \leq \frac{f(s_j)}{s_j} \leq v_o$.

Now we have for all j:

$$u - \varepsilon/3 < u_o = u - (2N-1)(v-u) < v_o = v + (2N-1)(v-u) < v + \varepsilon/3.$$

Therefore $u - \varepsilon/3 \le g(x) = \lim_{j\to+\infty} \frac{f(s_j)}{s_j} \le v + \varepsilon/3$ completing the proof.

We conclude with two remarks. First, while the condition of being balanced is sufficient to yield Baire 1 selective derivatives it is not necessary. It is easy to construct examples of functions which are selectively differentiable with respect to an unbalanced selection and are differentiable except at origin. Such selective derivatives are necessarily Baire class 1. It would be interesting to find both necessary and sufficient conditions. Second, it should be possible to translate the concept of balanced selective derivatives into a corresponding idea for path derivatives [2], this might yield a better approach to finding necessary and sufficient conditions.

References

[1] R. J. O'Malley, Selective Derivates, Acta Math. Acad. Sci. Hung. 29 (1-2) 1977, p. 77-97.

[2] A. M. Bruckner, B. Thomson, R. J. O'Malley, Path Derivatives, to appear in Trans. A.M.S.

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