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# ON NECESSARY AND SUFFICIENT CONDITIONS FOR NON-ABSOLUTE INTEGRABILITY

**Abstract**

We give necessary and sufficient conditions for a function to be general Denjoy or Kubota integrable. The conditions we obtain are in terms of a sequence of Lebesgue integrable functions converging to the given function.

In [1], Liu proved that if $f$ is Henstock integrable on $[a, b]$, then there exists an increasing sequence $\{X_n\}$ of closed sets whose union is $[a, b]$ such that $f$ satisfies condition $(L)$ on $\{X_n\}$. We recall that a function $f$ defined on $[a, b]$ is said to fulfill the condition $(L)$ on $\{X_n\}$ if $f$ is Lebesgue integrable on each $X_n$ and $(L)\int_{X_n \cap [a,x]} f(t)dt$ converge uniformly on $[a, b]$. For details, see [1]. In [2], the Controlled Convergence Theorem for the Kubota integral is given.

In this paper we shall also give a Riesz-type definition for the general Denjoy integral and the Kubota integral. The main result is:

**Theorem 1** *In order for $f$ to be Kubota integrable on $[a, b]$, it is necessary and sufficient that there exists an increasing sequence $\{X_n\}$ of closed sets whose union is $[a, b]$ such that*

*(i) $f$ is Lebesgue integrable on each $X_n$ ;*

*(ii) the sequence of primitives $\{F_n\}$ of $\{f\chi_{X_n}\}$ is $(UACG)([a, b])$ ;*

*(iii) $F_n \to F$ pointwise on $[a, b]$, where $F$ is approximately continuous on $[a, b]$.*

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Furthermore, we shall show that if $f$ is General Denjoy integrable on $[a, b]$, then conditions (i) and (iii) in the above theorem can be improved to

(iii)$'$ *$f$ satisfies condition (L) on $\{X_n\}$.*

First, we give some definitions.

Let $X \subseteq [a, b]$. A function $F$ defined on $[a, b]$ is said to be $AC(X)$ if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_k, b_k]\}$ with $a_k$ and $b_k$ belonging to $X$ for all k satisfying

$$\sum_k |b_k - a_k| < \eta \quad \text{we have} \quad \sum_k |F(b_k) - F(a_k)| < \varepsilon.$$

A function $F$ is said to be $(ACG)$ on $[a, b]$ if $[a, b]$ is the union of closed subsets $\{X_k\}$ such that the function $F$ is $AC(X_k)$ for each k.

A family of functions $\{F_n\}$ is said to be $UAC(X)$ if and only if every $F_n$ is $AC(X)$ and the $\eta > 0$ in the definition of $AC(X)$ are independent of n. A family of functions $\{F_n\}$ is said to be $(UACG)$ on $[a, b]$ if $\bigcup_{k=1}^{\infty} X_k = [a, b]$ with $\{F_n\}$ being $UAC(X_k)$ for each k.

A function $f$ defined on a compact interval $[a, b]$ is said to be Kubota integrable (respectively general Denjoy integrable) on $[a, b]$ if there is a function $F$ such that :

(i) $F$ is approximately continuous (respectively continuous) on $[a, b]$ ;

(ii) $F$ is $(ACG)([a, b])$;

(iii) the approximate derivative $F'_{ap}(x) = f(x)$ for almost all $x$ in $[a, b]$.

The function $F$ is called the primitive of $f$ on $[a, b]$. The integral of $f$ on $[a, b]$ is $F(b) - F(a)$, and we say that $f$ is integrable to $F(b) - F(a)$ on $[a, b]$. We write $F(b) - F(a) = (AD) \int_a^b f$ (respectively $F(b) - F(a) = (D) \int_a^b f$.) For more details, see [2], [3], [4], [5].

**Definition 1** *If $f$ is Kubota integrable on $[a, b]$, we shall say that $f$ has property (P) if the following conditions are satisfied :*

*Given a positive integer n, there is a sequence $\{X_n\}$ of closed sets in $[a, b]$ such that :*

*(1) $a, b \in X_1$, $X_n \subseteq X_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} X_n = [a, b]$.*

*(2) $f$ is Lebesgue integrable on $X_n$ for each n;*

*(3) for each $n$, if a finite sequence $\{I_i\}_{i=1}^{i_o}$ of non-overlapping intervals contained in $[a,b]$ satisfies the condition that both endpoints of each $I_i$ belong to $X_n$, then we have*

$$\left|\sum_{i=1}^{i_o}\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_n} f\right\}\right|<\frac{1}{n}.$$

We shall prove that if $f$ is Kubota integrable on $[a,b]$, then $f$ satisfies property $(P)$ on $[a,b]$. We shall apply the category proof. We need three Lemmas.

**Lemma 1** *(Cauchy extension) If $f$ is Kubota integrable on $[a,b]$, and $f$ has property (P) on $[a,x]$ for each $a<x<b$, then $f$ has property (P) on $[a,b]$.*

PROOF. Let $a=a_1<a_2<\cdots<b$ with $a_n\to b$ as $n\to\infty$ and

$$\left|(AD)\int_a^{a_n} f-(AD)\int_a^b f\right|<\frac{1}{2n} \tag{1}$$

Since $f$ has property $(P)$ on each of the interval $[a_k,a_{k+1}]$ for each positive integer k, there exists an increasing sequence $\{X_{k,n}\}_{n\ge1}$ of closed subsets of $[a_k,a_{k+1}]$ with $a_k,a_{k+1}\in X_{k,1}$ and $\bigcup_{n=1}^{\infty}X_{k,n}=[a_k,a_{k+1}]$ such that $f$ is Lebesgue integrable on each $X_{k,n}$ and if a finite sequence $\{I_i\}_{i=1}^{i_o}$ of non-overlapping intervals contained in $[a_k,a_{k+1}]$ satisfies the condition that both endpoints of each $I_i$ belong to $X_{k,n}$, then we have

$$\left|\sum_{i=1}^{i_o}\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_{k,n}} f\right\}\right|<\frac{1}{n2^{k+1}} \tag{2}$$

Put $X_n=\bigcup_{k=1}^{n}X_{k,n}\cup\{b\}$. Then $X_n\subseteq X_{n+1}$ for all n with $\bigcup_{n=1}^{\infty}X_n=[a,b]$. $f$ is Lebesgue integrable on each $X_n$. Now take a finite sequence of non-overlapping intervals $\{I_i\}_{i=1}^{s}$ in $[a,b]$ with both end points of each $I_i$ belonging to $X_n$. Note that $\bigcup_{k=1}^{n+1}\{a_k\}\subseteq X_n$, so we may suppose that $I_j\subseteq[a_l,a_{l+1}]$ for some $1\le l\le n$ if $I_j\cap(a_l,a_{l+1})$ is non-empty. Furthermore we may assume that $I_s=[a_{n+1},b]$. Note that $b\in X_1$ and $X_n\cap I_s=\{a_{n+1},b\}$. Thus we have

$$
\begin{aligned}
&\left|\sum_{i=1}^{s}\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_n} f\right\}\right| \\
=\ &\left|\sum_{i=1}^{s-1}\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_n} f\right\}+(AD)\int_{a_{n+1}}^{b} f\right| \\
\leq\ &\left|\sum_{i=1}^{s-1}\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_n} f\right\}\right|+\frac{1}{2(n+1)} \\
<\ &\left|\sum_{i=1}^{s-1}\left\{(AD)\int_{I_i} f-\sum_{k=1}^{n}(L)\int_{I_i\cap X_{k,n}} f\right\}\right|+\frac{1}{2n} \\
\leq\ &\sum_{k=1}^{n}\frac{1}{n2^{k+1}}+\frac{1}{2n}
\end{aligned}
$$

Since both endpoints of $I_i$ belong to $X_{k,n}$ if $I_i \subseteq [a_k, a_{k+1}]$ for some $k$, $1 \leq k \leq n$. Thus $f$ satisfies property $(P)$ on $[a,b]$. □

**Lemma 2** *If $f$ is Kubota integrable on $[a,b]$ with its primitive $F$ being $AC(X)$, where $X$ is a closed subset of $[a,b]$ with $a,b \in X$ and $(a,b)-X = \bigcup_{k=1}^{\infty}(a_k,b_k)$, then $f\chi_X$ is Lebesgue integrable on $[a,b]$ with*

$$(AD)\int_a^b f = (L)\int_a^b f\chi_X + \sum_{k=1}^{\infty}(AD)\int_{a_k}^{b_k} f.$$

PROOF. We may suppose that $a,b \in X$ with $(a,b)-X = \bigcup_{k=1}^{\infty}(a_k,b_k)$, for otherwise we may replace $[a,b]$ by the smallest closed interval containing $X$. Since $F$ is $AC(X)$ and so $F$ is $VB(X)$, the series $\sum_{k=1}^{\infty}\left|(AD)\int_{a_k}^{b_k} f\right|$ converges, and the function $H$ given by

$$H(x) = \sum_{k=1}^{\infty}(AD)\int_{a_k}^{b_k} f\chi_{[a,x]}$$

where $x \in [a,b]$, is well defined.

**Claim.** $H$ is approximately continuous on $[a,b]$.

PROOF OF CLAIM. We shall first prove that if $x \in [a,b)$, then $H$ is right approximately continuous at $x$, that is, given $\varepsilon > 0$, there exists an approximate neighborhood $D'_x$ of $x$ such that whenever $y \in D'_x$ with $x < y$, we have

$$|H(x) - H(y)| < \varepsilon. \tag{3}$$

If $x \in \bigcup_{k=1}^{\infty}[a_k, b_k)$, then (3) follows from the approximate continuity of $F$ on $[a, b]$. Suppose $x$ is a limit point of $X \bigcap [x, b]$. By the fact that $F$ is $AC(X)$, $F$ is continuous on $X$, there exists $\delta(x) > 0$ such that whenever $x < t < x + \delta(x)$ with $t \in X$, we have

$$|F(x) - F(t)| < \frac{\varepsilon}{3}. \tag{4}$$

Now we choose a positive integer $N$ so that

$$\sum_{k=N+1}^{\infty} \left|(AD)\int_{a_k}^{b_k} f\right| < \frac{\varepsilon}{3} \tag{5}$$

and we modify $\delta(x) > 0$, if necessary, so that $(x, x + \delta(x)) \cap [a_k, b_k] = \emptyset$ for $k = 1, 2, ..., N$. By the approximate continuity of $F$ at $x$, there exists an approximate neighborhood $D''_x$ of $x$ such that whenever $y \in D''_x$, we have

$$|F(x) - F(y)| < \frac{\varepsilon}{3}. \tag{6}$$

Put $D_x = D''_x \cap [x, x + \frac{1}{2}\delta(x)]$. Then whenever $y \in D_x - X$, $y \in (a_h, b_h)$ for some $h \geq N + 1$, and we have

$$\begin{aligned} & |H(x) - H(y)| \\ \leq\ & \sum_{k=1}^{\infty} \left|(AD)\int_{a_k}^{b_k} f\chi_{[x,y]}\right| \\ \leq\ & \sum_{k=N+1}^{\infty} \left|(AD)\int_{a_k}^{b_k} f\right| + \left|(AD)\int_{a_h}^{y} f\right| \end{aligned} \tag{7}$$

noting that $[x, y] \cap (a_k, b_k) = \emptyset$ for each $k = 1, \ldots, N$. If $y \in X$, then the term '$(AD)\int_{a_h}^{y} f$' disappears. By (7), for all $y \in D_x$ with $x < y$, we have

$$\begin{aligned} & |H(x) - H(y)| \\ \leq\ & \sum_{k=N+1}^{\infty} \left|(AD)\int_{a_k}^{b_k} f\right| + \left|(AD)\int_{a_h}^{y} f\right| \\ <\ & \frac{\varepsilon}{3} + \left|(AD)\int_{x}^{y} f\right| + \left|(AD)\int_{x}^{a_h} f\right|, \text{ by (5)}, \\ <\ & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}, \text{ by (6) and (4)}, \\ =\ & \varepsilon. \end{aligned}$$

Similarly, for $a < x \le b$, we can prove that $H$ is left approximately continuous at $x$. The proof of the claim is complete.

Put $G(a) = F(a), G(b) = F(b)$ and $G(x) = F(x)$, if $x \in X$ and

$$G(x) = \begin{cases} F(x), & \text{if } x \in X, \\ \frac{F(b_k)-F(a_k)}{b_k-a_k}(x-a_k) + F(a_k), & \text{if } x \in (a_k, b_k). \end{cases}$$

Then $G$ is absolutely continuous on $[a,b]$, and $G'(x) = F'_{ap}(x) = f(x)$ for almost all $x \in X$. Hence $f$ is Lebesgue integrable on $X$. It is obvious that $H$ is $AC(X)$ and that $H$ is $(ACG)$ on $[a,b]$.

Put

$$h_0(x) = \begin{cases} 0, & \text{if } x \in X \\ \frac{F(b_k)-F(a_k)}{b_k-a_k}, & \text{if } x \in (a_k, b_k). \end{cases}$$

Then $h_0$ is Lebesgue integrable on $[a,b]$ with primitive $H_0$ (say). Furthermore, $H(x) = H_0(x)$ for all $x \in X$. Hence $H'_{ap}(x) = H'_0(x) = h_0(x) = 0$ for almost all $x \in X$. Furthermore, for each positive integer k, $H'_{ap}(x) = f(x)$ for almost all $x \in (a_k, b_k)$. By the above claim, the function $H$ is approximately continuous on $[a,b]$ and $H$ is $(ACG)$ on $[a,b]$, the function $h$ defined on $[a,b]$ by

$$h(x) = \begin{cases} 0, & \text{if } x \in X, \\ f(x), & \text{if } x \notin X, \end{cases}$$

is Kubota integrable on $[a,b]$ with primitive $H$. Hence we have

$$(AD)\int_a^b f = (L)\int_a^b f\chi_X + (AD)\int_a^b h = (L)\int_a^b f\chi_X + \sum_{k=1}^{\infty}(AD)\int_{a_k}^{b_k} f.$$

The proof is complete. □

**Lemma 3** *(Harnack extension). Let the hypothesis be as in Lemma 2, and if $f$ has property (P) on each of the interval $[a_k, b_k]$, then $f$ has property (P) on $[a,b]$.*

PROOF. Since $f$ has property $(P)$ on each of the interval $[a_k, b_k]$ for each positive integer k, there exists an increasing sequence $\{X_{k,n}\}_{n\ge 1}$ of closed subsets of $[a_k, b_k]$ with $a_k, b_k \in X_{k,1}$ and $\bigcup_{n=1}^{\infty} X_{k,n} = [a_k, b_k]$ such that $f$ is Lebesgue integrable on each $X_{k,n}$ and if a finite sequence $\{I_i\}_{i=1}^{i_0}$ of non-overlapping intervals contained in $[a_k, b_k]$ satisfies the condition that both endpoints of each $I_i$ belong to $X_{k,n}$, then we have

$$\left|\sum_{i=1}^{i_0}\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_{k,n}} f\right\}\right| < \frac{1}{n2^{k+1}} \tag{8}$$

Since $F$ is $AC(X)$, by Lemma 2, $f$ is Lebesgue integrable on $X$.

Since the series $\sum\limits_{k=1}^{\infty}\left|(AD)\int_{a_k}^{b_k} f\right|$ converges. For each positive integer n, we choose a positive integer $k(n)$ such that

$$\sum_{k=k(n)+1}^{\infty}\left|(AD)\int_{a_k}^{b_k} f\right| < \frac{1}{2n} \tag{9}$$

and we may suppose that $k(n+1) > k(n)$ for all positive integer n.

Put $X_n = \bigcup\limits_{k=1}^{k(n)} X_{k,n} \cup X$. Then $X_n \subseteq X_{n+1}$ for all n with $\bigcup\limits_{n=1}^{\infty} X_n = [a,b]$. $f$ is Lebesgue integrable on each $X_n$. Now take a finite sequence of non-overlapping intervals $\{I_i\}_{i=1}^{s}$ in $[a,b]$ with both end points of each $I_i$ belonging to $X_n$. Since $\bigcup\limits_{k=1}^{k(n)}(\{a_k\}\cup\{b_k\}) \subseteq X \cap \bigcup\limits_{k=1}^{k(n)} X_{k,n}$, we may assume that $I_i \subseteq [a_k, b_k]$ if $I_i \cap (a_k, b_k)$ is non-empty for some $1 \le k \le k(n)$. Note that if $I_j \cap (a_k, b_k)$ is empty for all $k = 1, 2, \ldots, k(n)$, then both end points of $I_j$ belong to $X$. Split the sum $\sum_{i=1}^{s}\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_n} f\right\}$ into two partial sums $\sum\limits_1$ and $\sum\limits_2$. In the sum $\sum\limits_1$, each interval $I_i$ involved belongs to $[a_k, b_k]$ for some $1 \le k \le k(n)$, and $\sum\limits_2 = \sum - \sum\limits_1$. Then we have

$$\begin{aligned}
&\left|\sum_{i=1}^{s}\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_n} f\right\}\right| \\
\le\ & \left|\sum_1\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_n} f\right\}\right| \\
+\ & \sum_2\left|\left\{(AD)\int_{I_i} f-(L)\int_{I_i\cap X_n} f\right\}\right| \\
\le\ & \sum_{k=1}^{k(n)}\frac{1}{n2^{k+1}} + \sum_{k=k(n)+1}^{\infty}\left|(AD)\int_{a_k}^{b_k} f\right| \quad \text{by (8) and Lemma 2} \\
<\ & \frac{1}{2n}+\frac{1}{2n} \quad \text{by (9)},
\end{aligned}$$

and so $f$ satisfies property $(P)$ on $[a,b]$. The proof is complete. □

**Theorem 2** *If $f$ is Kubota integrable on $[a,b]$ with primitive $F$, then $f$ satisfies property $(P)$ on $[a,b]$.*

**PROOF.** We say that an interval $I \subseteq [a,b]$ is regular if the function $f$ satisfies property $(P)$ on $I$. Furthermore, we say that a point $x \in [a,b]$ is regular if each sufficiently small interval $I \subseteq [a,b]$ containing $x$ as an interior point is regular. Let $P_1$ be the set of all non-regular points in $[a,b]$. Then the set $P_1$ is closed and every subinterval of $[a,b]$ which contains no points of this set is regular. In view of $F$ being $(ACG)$ on $[a,b]$ and the Baire's category Theorem, f is Lebesgue integrable on some interval in $[a,b]$. By dominated convergence theorem, we see that the set of regular points is non-empty.

Suppose, if possible, $P_1 \neq \emptyset$. By Lemma 1, we see easily that every interval contagious to $P_1$ is regular and that the set $P_1$ therefore has no isolated points. Again, by the Baire's category Theorem, there is a portion $P_o$ of $P_1$ such that $F$ is $AC(P_o)$. Let $J_o$ be the smallest closed interval containing $P_o$. Since the set $P_1$ contains no isolated points, the same is true of any portion of $P_1$, and therefore $P_1 \cap J_o$ has non-empty interior. It follows that in order to obtain a contradiction, we need only to prove that the interval $J_o$ is regular.

To show this, let $Q$ be the set consisting of the points of the set $P_1 \cap J_o$ and of the end-points of $J_o$. We denote by $\{I_n\}$ the sequence of the intervals contiguous to $Q$. Now $F$ is $AC(Q)$ and $f$ satisfies property $(P)$ on each of the intervals $I_n$. By Lemma 3, $f$ satisfies property $(P)$ on $J_o$, the interval $J_o$ is regular and this completes the proof. □

Proof of the main theorem. The sufficiency follows from the Controlled Convergence Theorem in [2]. Now, we shall prove the necessity.

**PROOF.** By Theorem 1, f satisfies property $(P)$ on $[a,b]$. Let $\{X_n\}$ be the sequence of closed sets given as in Theorem 1. Then $f$ is Lebesgue integrable on each $X_n$.

**Claim.** $F$ is $AC(X_p)$ for each positive integer p.

Let $\varepsilon > 0$ be given. Choose an integer $N > p$ so that $\frac{2}{N} < \frac{\varepsilon}{2}$. By property $(P)$, whenever $\{[u_i, v_i]\}$ is a finite sequence of non-overlapping intervals with both $u_i, v_i \in X_p$ for each $i$, we have

$$\left|\sum \left\{(AD)\int_{u_i}^{v_i} f - (L)\int_{u_i}^{v_i} f\chi_{X_N}\right\}\right| < \frac{1}{N} \tag{10}$$

since $X_p \subseteq X_N$ for $N > p$. Now the sequence $\{[u_i, v_i]\}$ is arbitrary, so by (10),

$$\sum \left|(AD)\int_{u_i}^{v_i} f - (L)\int_{u_i}^{v_i} f\chi_{X_N}\right| < \frac{2}{N} \tag{11}$$

Since $f$ is Lebesgue integrable on each $X_n$, there exists $\eta_n^* > 0$ such that whenever $\{[u_j, v_j]\}$ is a finite sequence of non-overlapping intervals in $[a,b]$ satisfying $\sum_j (v_j - u_j) < \eta_n^*$, we have

$$\sum_j \left| (L) \int_{u_j}^{v_j} f\chi_{X_n} \right| < \frac{\varepsilon}{2} \text{ for each n} \tag{12}$$

Consequently, by (11), (12) and the fact that $\frac{2}{N} < \frac{\varepsilon}{2}$, whenever $\{[u_k, v_k]\}$ is a finite sequence of non-overlapping intervals in $[a,b]$ with $u_k, v_k \in X_p$ for each k satisfying

$$\sum_k (v_k - u_k) < \eta_N^* \quad \text{we have} \quad \sum_k \left| (AD) \int_{u_k}^{v_k} f \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{13}$$

and so $F$ is $AC(X_p)$ for each p.

Put $F_n(x) = (L)\int_a^x f\chi_{X_n}$ for each $x \in [a,b]$. We claim that $\{F_n\}$ is $UAC(X_p)$ for each p.

By property $(P)$, whenever $\{[u_i, v_i]\}$ is a finite sequence of non-overlapping intervals with both $u_i, v_i \in X_n$ for each $i$, we have

$$\left| \sum \left\{ (AD) \int_{u_i}^{v_i} f - (L) \int_{u_i}^{v_i} f\chi_{X_n} \right\} \right| < \frac{1}{n} \tag{14}$$

By (14), we have

$$\sum \left| (AD) \int_{u_i}^{v_i} f - (L) \int_{u_i}^{v_i} f\chi_{X_n} \right| < \frac{2}{n} \tag{15}$$

Consequently, whenever $\{[u_k, v_k]\}$ is a finite sequence of non-overlapping intervals in $[a,b]$ with $u_k, v_k \in X_p$ for each k satisfying

$$\sum_k (v_k - u_k) < \min\left\{ \eta_1^*, \eta_2^*, \ldots, \eta_N^* \right\},$$

we have

$$\sum_k \left| (L) \int_{u_k}^{v_k} f\chi_{X_n} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for all n}$$

by (13) and (15). Hence $\{F_n\}$ is $UAC(X_p)$ for each p.

To prove that $F_n \to F$ pointwise on $[a,b]$, let $x \in (a,b]$ be given. Then $a, x \in X_{N_0}$ for some positive integer $N_0 = N_0(x, \varepsilon)$ such that $\frac{1}{N_0} < \varepsilon$. By property $(P)$, for all $n \geq N_0$,

$$\left|(L)\int_a^x f\chi_{X_n} - (AD)\int_a^x f\right| < \frac{1}{n} \le \frac{1}{N_0} < \varepsilon.$$

The proof is complete. □

**Remark 1** *By using the main Theorem, we can easily give a Riesz-type definition for the Kubota integral.*

**Corollary 1** *In order for $f$ to be general Denjoy integrable on $[a,b]$ with primitive $F$, it is necessary and sufficient that there exists an increasing sequence $\{X_n\}$ of closed sets whose union is $[a,b]$ such that*

*(i) $f$ is Lebesgue integrable on each $X_n$ ;*

*(ii) the sequence of primitives $\{F_n\}$ of $\{f\chi_{X_n}\}$ is $(UACG)([a,b])$ ;*

*(iii) $(iii)'$ $F_n$ converge uniformly on $[a,b]$ to $F$.*

PROOF. Again the sufficiency follows from the Controlled Convergence Theorem in [2]. Now we shall prove the necessity. Since $f$ is general Denjoy integrable on $[a,b]$, $f$ is also Kubota integrable on $[a,b]$ with primitive $F$. By Theorem 1, conditions (i) and (ii) are satisfied. It remains to prove (iii)$'$ holds.

Since $F$ is uniformly continuous on $[a,b]$, given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in [a,b]$ with $|x-y| < \delta$, we have

$$|F(x) - F(y)| < \frac{\varepsilon}{2} \tag{16}$$

Since $\bigcup_{n=1}^{\infty} X_n = [a,b]$ with $X_n \subseteq X_{n+1}$ for all n, we may choose a positive integer $N^*$ such that

$$|[a,b] - X_n| < \delta \quad \text{for all } n \ge N^* \text{ and } \frac{1}{N^*} < \frac{\varepsilon}{2} \tag{17}$$

For $n \ge N^*$, by property $(P)$, if $x \in X_n$, then by (17),

$$\left|(L)\int_a^x f\chi_{X_n} - (D)\int_a^x f\right| < \frac{1}{n} \le \frac{1}{N^*} < \frac{\varepsilon}{2} \tag{18}$$

If $x \notin X_n$, then $x \in (y,z)$ for some $y, z \in X_n$ with $(y,z) \subseteq (a,b) - X_n$. Thus $|x-y| \le |(a,b) - X_n| \le |(a,b) - X_{N^*}| < \delta$. Hence we have, by (16) and (18),

$$
\begin{aligned}
&\left|(L)\int_a^x f\chi_{X_n} - (D)\int_a^x f\right| \\
\le\ &\left|(L)\int_a^y f\chi_{X_n} - (D)\int_a^y f\right| + \left|(D)\int_y^x f\right| \\
<\ &\frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
=\ &\varepsilon
\end{aligned}
$$

Thus

$$\sup_{a\le x\le b}\left|(L)\int_a^x f\chi_{X_n} - (D)\int_a^x f\right| \le \varepsilon \quad \text{for all } n \ge N^*,$$

that is, (iii)′ holds. The proof is complete. □

**Remark 2** *Conditions (i) and* (iii)′ *of the Corollary says that $f$ satisfies condition $(L)$ on $\{X_n\}$.*

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