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MEASURABILITY, QUASICONTINUITY AND CLIQUISHNESS OF FUNCTIONS OF TWO VARIABLES

Abstract

Several properties (measurability, quasicontinuity and cliquishness) of functions of two variables having special sections are proved.

Denote by $\mathbb{R}$ the set of all reals and by $\mathbb{R}^2$ the product space $\mathbb{R} \times \mathbb{R}$. Let (X, τ) be a topological space with a topology τ. A function $f : X \to \mathbb{R}$ is said to be τ quasicontinuous (τ cliquish) at a point $x \in X$ if for every set $U \in \tau$ containing x and for every $\eta > 0$ there is a nonempty set $V \in \tau$ such that $V \subset U$ and $|f(t) - f(x)| < \eta$ for every $t \in V$ ($\operatorname{osc}_V f < \eta$) [12] .

A point $x \in \mathbb{R}$ is said to be a density point of a measurable (in the sense of Lebesgue) set $A \subset \mathbb{R}$ if $\lim_{h \to 0} \mu([x-h, x+h] \cap A)/2h = 1$ where μ denotes Lebesgue measure.

The family

$$\tau_d = \{A \subset \mathbb{R}; A \text{ is measurable and every point } x \in A \text{ is a density point of } A\}$$

is a topology called the density topology [1].

Analogously, the family $\tau_{ae} = \{A \in \tau_d; \mu(A \setminus \operatorname{int} A) = 0\}$, where $\operatorname{int} A$ denotes the Euclidean interior of A , is a topology called *a.e.* topology (O'Malley [9]).

For a given topological space (X, τ) we define the following condition:

(1) there is a countable subfamily $\Phi \subset \tau$ such that for every nonempty set $U \in \tau$ there is a set $V \in \Phi$ such that $V \subset U$ and $V \neq \emptyset$.

Observe that the topology τ_{ae} satisfies condition (1).

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Theorem 0.1 *Assume that a topological space* (X, τ_X) *satisfies condition* (1) *and a topological space* (Y, τ_Y) *is a Baire space. If a function* $f : X \times Y \to \mathbb{R}$ *is such that all sections*

$$f_x(v) = f(x, v), x \in X, v \in Y$$

and

$$f^y(u) = f(u, y), u \in X, y \in Y$$

are τ_Y *and respectively* τ_X *quasicontinuous, then* f *is* $(\tau_X \times \tau_Y)$ *quasicontinuous.*

PROOF. Fix a positive real η, a point $(x, y) \in X \times Y$ and a set $W \in \tau_X \times \tau_Y$ containing the point (x, y). Let $U \in \tau_X$ and $V \in \tau_Y$ be such that $x \in U, y \in V$ and $U \times V \subset W$. Let $I_1, \ldots, I_n, \ldots$ be a subfamily of nonempty sets of τ_X such that for every nonempty set $Z \in \tau_X$ there is a set $I_n \subset Z$. Since all sections $f^v, v \in V$, are τ_X quasicontinuous, for each $v \in V$ there is an index $n(v)$ such that $I_{n(v)} \subset U$ and for every $u \in I_{n(v)}$ we have $|f(x, v) - f(u, v)| < \eta/4$.

Similarly, by the τ_Y quasicontinuity of the section f_x at a point y there is a nonempty set $L \subset V$ such that $L \in \tau_Y$ and for all $v \in L$ we have

$$|f(x, v) - f(x, y)| < \eta/4.$$

But the set L is of the second category, so there is an index n_1 such that the set $A = \{v \in L; n(v) = n_1\}$ is of the second category. Let a set $M \subset L$ be such that $\emptyset \neq M \in \tau_Y, M \subset \operatorname{cl} A$, where $\operatorname{cl} A$ denotes the closure (in τ_Y) of the set A. Then the set $K = I_{n_1} \times M \in \tau_X \times \tau_Y$ is nonempty and $K \subset W$. For each $(u, v) \in I_{n_1} \times A$ we have

$$|f(u, v) - f(x, y)| \leq |f(u, v) - f(x, v)| + |f(x, v) - f(x, y)| < \eta/4 + \eta/4 = \eta/2.$$

Let $(t, w) \in K$. Since the section f_t is τ_Y quasicontinuous at w and since the set $A \cap M$ is dense in M, there is a point $z \in A \cap M$ such that $|f(t, z) - f(t, w)| < \eta/4$. Consequently,

$$|f(t, w) - f(x, y)| \leq |f(t, w) - f(t, z)| + |f(t, z) + f(x, y)| < \eta/4 + \eta/2 < \eta.$$

$\square$

Remark 0.1 *The method of the proof of Theorem 0.1 is known and used e.g. in [11].*

Theorem 0.2 *Let the spaces* (X, τ_X) *and* (Y, τ_Y) *be the same as in Theorem 0.1. If all sections* f_x *of a function* $f : X \times Y \to \mathbb{R}$ *are* τ_Y *quasicontinuous and if all sections* f^y *are* τ_X *cliquish, then* f *is* $(\tau_X \times \tau_Y)$ *cliquish.*

PROOF. Adopt the notation from the proof of Theorem 0.1. Since all sections $f^v, v \in V$, are τ_X cliquish, for each $v \in V$ there is an index $k(v)$ such that $I_{k(v)} \subset U$ and $\mathrm{osc}_{I_{k(v)}} f_v < \eta/8$. There is an index k_1 such that the set $B = \{v \in L; k(v) = k_1\}$ is of the second category. Let $D \subset \mathrm{cl}\, B$ be a nonempty set belonging to τ_Y and let $(u, v) \in I_{k_1} \times (D \cap B)$ be a fixed point. Since the section f_u is τ_Y quasicontinuous at v, there is a nonempty set $T \in \tau_Y$ such that $T \subset D$ and for each point $w \in T$ we have $|f(u, w) - f(u, v)| < \eta/4$. Let $N = I_{k_1} \times T$. Then $N \subset W$ is a nonempty set belonging to $\tau_X \times \tau_Y$. For each point $(t, w) \in I_{k_1} \times (T \cap B)$ we have

$$\begin{aligned}|f(t,w) - f(u,v)| &\leq |f(t,w) - f(u,w)| + |f(u,w) - f(u,v)| \\ &< \eta/8 + \eta/4 < 3\eta/8.\end{aligned}$$

If a point $(s, z) \in N$, then there is a point $w \in T \cap B$ such that $|f(s, w) - f(s, z)| < \eta/8$ and consequently,

$$\begin{aligned}|f(s,z) - f(u,v)| &\leq |f(s,z) - f(s,w)| + |f(s,w) - f(u,v)| \\ &< \eta/8 + 3\eta/8 = \eta/2.\end{aligned}$$

This proves that $\mathrm{osc}_N f \leq \eta$.

Since $(\mathbb{R}, \tau_{ae})$ is a Baire space satisfying condition (1), we have the following consequence.

Corollary 0.1 *If all sections f_x of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are τ_{ae} quasicontinuous and all sections f^y are τ_{ae} quasicontinuous (τ_{ae} cliquish), then f is $(\tau_{ae} \times \tau_{ae})$ quasicontinuous ($(\tau_{ae} \times \tau_{ae})$ cliquish).*

Remark 0.2 *Observe that the topology τ_{ae} does not satisfy the second countability axiom in a nonempty set $U \in \tau_{ae}$. So, Corollary 0.1 is not a direct consequence of Theorem 3 in [3] and Theorem 4.1.2 in [12].*

Let τ_e denote the Euclidean topology in $\mathbb{R}$. For a topological space (X, τ) let

$$Q(\tau) = \{f : X \to \mathbb{R}; f \text{ is } \tau \text{ quasicontinuous}\}$$

and

$$P(\tau) = \{f : X \to \mathbb{R}; f \text{ is } \tau \text{ cliquish}\}.$$

We have ([4]) $Q(\tau_{ae}) \subset Q(\tau_e) \subset P(\tau_e) = P(\tau_{ae})$, where all inclusions are proper.

Lemma 0.1 *Let $C \subset \mathbb{R}$ be a nonempty nowhere dense perfect set. If a function $f : \mathbb{R} \to [0,1]$ is such that for every component I of the set $\mathbb{R} \setminus C$ the restricted function $f|\operatorname{cl} I$ is τ_e continuous and $f(\operatorname{cl} I) = [0,1]$, then f is τ_{ae} quasicontinuous at each point $x \in C$ which is not a density point of $\mathbb{R} \setminus C$.*

PROOF. Let $x \in C$ be a point which is not a density point of $\mathbb{R} \setminus C$, and let $U \in \tau_{ae}$ be a set containing x. Then $\mu(C \cap U) > 0$ and there is a component I of $\mathbb{R} \setminus C$ which is contained in $\operatorname{int} U$. Fix a positive η. Since $f(x) \in [0,1]$ and $f(\operatorname{cl} I) = [0,1]$, there is an open interval $J \subset I$ such that $f(J) \subset (f(x) - \eta, f(x) + \eta)$. But $J \in \tau_e \subset \tau_{ae}$. □

Corollary 0.2 *There is a τ_{ae} quasicontinuous function $f : \mathbb{R} \to [0,1]$ which is not measurable (in the sense of Lebesgue).*

PROOF. Let $C \subset (0,1)$ be a Cantor set of positive measure which does not contain density points of the set $\mathbb{R} \setminus C$ and let

$$B = \{x \in C; x \text{ is a density point of } C\}.$$

If $A \subset B$ is a nonmeasurable set, then each function $f : \mathbb{R} \to [0,1]$ such that $f(x) = 1$ for $x \in A$, $f(x) = 0$ for $x \in C \setminus A$ and for every component I of the set $\mathbb{R} \setminus C$, the restricted function $f|\operatorname{cl} I$ is continuous and $f(\operatorname{cl} I) = [0,1]$, is τ_{ae} quasicontinuous, by Lemma 0.1. Evidently, f is not measurable. □

It is known that there are nonmeasurable functions $f : \mathbb{R}^2 \to [0,1]$ with measurable sections f_x and τ_d continuous sections f^y ([5]). Moreover, if all sections f_x of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are measurable and all its sections f^y are τ_{ae} continuous, then f is measurable ([5]).

Theorem 0.3 *Assume Martin's Axiom* **(MA)**. *There is a nonmeasurable function $f : \mathbb{R}^2 \to [0,1]$ such that all its sections f_x are τ_d quasicontinuous (and hence measurable) and all its sections f^y are measurable and τ_{ae} quasicontinuous.*

PROOF. There is a nonmeasurable function $g : \mathbb{R}^2 \to [0,1]$ such that all its sections f_x and f^y are τ_d quasicontinuous ([8]). There is also a Cantor set $C \subset \mathbb{R}$ of positive measure such that the restricted function $g|(C \times \mathbb{R})$ is not measurable. Let $B = \{x \in C; x \text{ is a density point of } C\}$ and let $h : \mathbb{R} \to [0,1]$ be a τ_{ae} quasicontinuous function such that for every component I of $\mathbb{R} \setminus C$ the restricted function $h|\operatorname{cl} I$ is τ_e continuous and $h(\operatorname{cl} I) = [0,1]$. Set

$$f(x,y) = \begin{cases} g(x,y) & \text{if } x \in B \\ h(x) & \text{if } x \in \mathbb{R} \setminus B. \end{cases}$$

Evidently, f is not measurable and all its sections f_x are τ_d quasicontinuous. Since all functions τ_d quasicontinuous are measurable ([8]), all sections f^y are measurable, and by Lemma 0.1, they are τ_{ae} quasicontinuous. □

Remark 0.3 *Observe that from its proof the function of Theorem 0.3 is* $(\tau_{ae} \times \tau_d)$ *quasicontinuous.*

Now we introduce the following definitions:

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strongly τ_d quasicontinuous (strongly τ_d cliquish) at a point $x \in \mathbb{R}$ if for every $\eta > 0$ and for every set $U \in \tau_d$ such that $x \in U$ there is an open interval I such that $I \cap U \neq \emptyset$ and for every $t \in I \cap U$ we have $|f(t) - f(x)| < \eta$ (resp. $I\, \mathrm{osc}_{I\cap U} f \leq \eta$).

Remark 0.4 *The definition of the strong* τ_d *cliquishness of a function* $f : \mathbb{R} \to \mathbb{R}$ *is equivalent to property* (G) *introduced in [5].*

Remark 0.5 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is strongly* τ_d *quasicontinuous (strongly* τ_d *cliquish) at a point* $x \in \mathbb{R}$ *if and only if for every* $\eta > 0$ *and for every* F_σ *set* $U \in \tau_d$ *containing* x *there is an open interval* I *such that* $I \cap U \neq \emptyset$ *and for each* $t \in I \cap U$ *we have* $|f(t) - f(x)| < \eta$ *(resp.* $\mathrm{osc}_{I\cap U} f \leq \eta$*).*

PROOF. Indeed if $U \in \tau_d$ is a set containing x, then there is an F_σ set $W \subset U$ belonging to τ_d such that $\mathrm{cl}(\{(t, f(t)); t \in W\}) = \mathrm{cl}(\{(t, f(t)); t \in U\})$. If I is an open interval such that $I \cap W \neq \emptyset$ and $|f(t) - f(x)| < \eta/2$ for $t \in I \cap W$, then $|f(t) - f(x)| \leq \eta/2 < \eta$ for $t \in I \cap U$. Similarly, if $\mathrm{osc}_{I\cap W} f \leq \eta$, then $\mathrm{osc}_{I\cap U} f \leq \eta$. □

Remark 0.6 *Evidently every* τ_{ae} *continuous function* $f : \mathbb{R} \to \mathbb{R}$ *is strongly* τ_d *quasicontinuous. However there is a* τ_{ae} *quasicontinuous function* $f : \mathbb{R} \to \mathbb{R}$ *which is not strongly* τ_d *quasicontinuous and which is such that for each countable union* A *of perfect sets the restricted function* $f|A$ *has a continuity point. (Such functions were introduced by Peek in [13].)*

PROOF. For example, let $C \subset (0,1)$ be a Cantor set of positive measure and let $\{I_n\}$ be a sequence of all components of the set $(0,1) \setminus C$. For each $n = 1, 2, \ldots$ we find a closed interval J_n having the same center as I_n and such that $\mu(J_n) < \mu(I_n)/n$ and a closed interval $K_n \subset \mathrm{int}\, J_n$. Then every function $f : \mathbb{R} \to \mathbb{R}$ such that for every $n = 1, 2, \ldots$, the restricted function $f|\,\mathrm{cl}\, I_n$ is τ_e continuous, $f(K_n) = \{1\}$, $f(J_n) = [0,1]$ and such that $f(x) = 0$ for $x \in \mathbb{R} \setminus \bigcup_{n\geq 1} \mathrm{int}\, J_n$ satisfies all required conditions. Indeed if $x \in C$ is a density point of C and $D \subset C$ is a set belonging to τ_d and containing x, then $E = D \cup \bigcup_{n\geq 1} \mathrm{int}\, K_n$ is a τ_d-neighborhood of x and for each open interval I such that $E \cap I \neq \emptyset$ there is an index n_1 with $\mathrm{int}\, K_{n_1} \cap I \neq \emptyset$. Since $f(t) = 1$ for

$t \in K_{n_1}$ and since $f(x) = 0$, the function f is not strongly τ_d quasicontinuous at x. Obviously f is pointwise discontinuous on every countable union of perfect sets and, by Lemma 0.1, f is τ_{ae} quasicontinuous. □

Remark 0.7 *Since all strongly τ_d quasicontinuous function are quasicontinuous and almost everywhere continuous [6], every function $f : \mathbb{R}^2 \to \mathbb{R}$ having all sections f_x strongly τ_d quasicontinuous and all sections f^y measurable is measurable [10].*

The functions $g_t : \mathbb{R} \to \mathbb{R}$ of a family $\{g_t\}_{t \in S}$, where S is a set of indexes, is said to be strongly upper τ_d equiquasicontinuous at a point x if for every $\eta > 0$ and for every set $U \in \tau_d$ containing x there is an open interval I such that $I \cap U \neq \emptyset$ and $f_t(u) - f_t(x) < \eta$ for every $u \in I \cap U$ and $t \in S$.

Theorem 0.4 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that all sections f_x are strongly upper τ_d equiquasicontinuous and all sections f^y are measurable. Then f is measurable.*

PROOF. By Lemma 2 in [2] it suffices to prove that for every $\eta > 0$ and for every measurable set $A \subset \mathbb{R}^2$ of positive measure there is a measurable set $B \subset A$ of positive measure such that $\operatorname{osc}_B f \leq \eta$. Without loss of generality we may suppose that f is bounded below, since if all functions $\max(a, f), a \in \mathbb{R}$, are measurable, then f is also measurable and all functions $\max(a, f)$ satisfy the hypothesis of our Theorem.

Fix a real $\eta > 0$ and a measurable set $A \subset \mathbb{R}^2$ of positive measure. Let

$$a = \operatorname{ess\,inf}_A f \doteq \sup\{\inf_{A \setminus C} f; \mu_2(C) = 0\},$$

where μ_2 denotes Lebesgue measure in $\mathbb{R}^2$. Let $A_1 = \{(x, y) \in A; f(x, y) \geq a\}$. There is a measurable set $D \subset A_1$ such that $\mu_2(A_1 \setminus D) = 0$ and if $(x, y) \in D$, then the section $D_x = \{v \in \mathbb{R}; (x, v) \in D\}$ is measurable and y is a density point of D_x ([14], pages 130–131). Since $\mu_2(A \setminus D) = 0$, we have $\operatorname{ess\,inf}_D f = a$. Let $E = \{(x, y) \in D; f(x, y) < a + \eta/4\}$. Then $E \subset D$ is of positive outer measure. From the strong upper τ_d equiquasicontinuity of all sections $f_x, x \in \mathbb{R}$, it follows that for every $(x, y) \in E$ there is an open interval $I(x, y)$ with rational endpoints such that $I(x, y) \cap D_x \neq \emptyset$ and $f(x, v) - f(x, y) < \eta/4$ for each $v \in I(x, y)$. Since the set E is of positive outer measure and the set of all intervals with rational endpoints is countable, there is an open interval I_1 such that the set $F = \{(x, y) \in E; I(x, y) = I_1\}$ is of positive outer measure, and consequently, its projection $\operatorname{pr}_x F = \{x \in \mathbb{R}; \exists_y (x, y) \in F\}$ is also of positive outer measure. Let $G \supset \operatorname{pr}_x F$ be a measurable cover of $\operatorname{pr}_x F$ (i.e., a measurable set such that if $C \subset G \setminus \operatorname{pr}_x F$ is measurable, then $\mu(C) = 0$).

Put $H = (G \times I_1) \cap D$ and observe that H is measurable. Since all sections H_x for $x \in \mathrm{pr}_x F$ are of positive measure, the set H is of positive measure. Fix $y_1 \in I_1$ such that the section $H^{y_1} = \{x; (x, y_1) \in H\}$ is measurable of positive measure. The section $x \to f(x, y_1), x \in \mathbb{R}$, is measurable. So there is a set $U \in \tau_d$ such that $U \subset H^{y_1}$ and $\mathrm{osc}_U f^{y_1} < \eta/4$. From the strong upper τ_d equiquasicontinuity of the sections f_x it follows that there is an open interval $I_2 \subset I_1$ such that for each $x \in H^{y_1}$ the set $I_2 \cap D_x$ is nonempty and $f(x, v) - f(x, y_1) < \eta/4$ for every $v \in I_2 \cap D_x$. Put $B = (H^{y_1} \times I_2) \cap D$. Since for each $x \in H^{y_1}$ the section B_x is of positive measure, the set B is of positive measure. Fix a point $x_1 \in H^{y_1} \cap \mathrm{pr}_x F$. For every $(x, y) \in B$ we have

$$\begin{aligned} a \leq f(x,y) &\leq f(x,y_1) + \eta/4 < f(x_1,y_1) + \eta/4 + \eta/4 \\ &< a + \eta/4 + \eta/4 + \eta/4 < a + 3\eta/4. \end{aligned}$$

So, $\mathrm{osc}_B f < \eta$. □

Theorem 0.5 *Let $f : \mathbb{R}^2 \to \mathbb{R}$. If all sections $f_x, x \in \mathbb{R}$, are strongly upper τ_d equiquasicontinuous and if all sections $f^y, y \in \mathbb{R}$, have the Baire property, then f has the Baire property.*

PROOF. The proof is similar as the proof of the previous theorem. As before we can suppose that f is bounded below. By Theorem 1 in [7] it suffices to prove that for every positive η and for every set $A \subset \mathbb{R}^2$ with the Baire property and of the second category there is a set $B \subset A$ with the Baire property and of the second category such that $\mathrm{osc}_B f \leq \eta$. Let $\eta > 0$ be a real number and let $A \subset \mathbb{R}^2$ be a set of the second category having the Baire property. Let A_1 be a nonempty open set such that $A_1 \setminus A$ is of the first category. There is a set $B_1 \subset \mathbb{R}$ of the first category such that if x is not in B_1, then the section $(A_1 \setminus A)_x$ is of the first category. Let

$$a = (B)\,\mathrm{ess\,inf}_{A_1} f = \sup\{\inf_{A_1 \setminus C} f; C \text{ is of the first category}\}$$

and let $A_2 = \{(x, y) \in A_1; f(x, y) \geq a\}$. Observe that $(B)\,\mathrm{ess\,sup}_{A_2} f = a$. Let $D = \{(x, y) \in A_2; f(x, y) < a + \eta/4\}$. Evidently D is of the second category. By the strong upper τ_d equiquasicontinuity of all sections $f_x, x \in \mathbb{R}$, for each point $(x, y) \in D$, there is an open interval $I(x, y)$ with rational endpoints such that $I(x, y) \cap D_x \neq \emptyset$ and $f(x, v) < a + \eta/4$ for each point $v \in D_x \cap I(x, y)$. There is an open interval I_1 such that $E = \{(x, y) \in D; I(x, y) = I_1\}$ is of the second category, and consequently, its projection $\mathrm{pr}_x E$ is also of the second category. Let $F \supset \mathrm{pr}_x E$ be a Baire cover of $\mathrm{pr}_x E$ (i.e., a set with the Baire property such that if a set $C \subset F \setminus \mathrm{pr}_x E$ has the Baire property, then C is of the first category). Put $G = (F \times I_1) \cap D$ and let $y_1 \in I_1$ be a point such that the section G^{y_1} has the Baire property and is of the second category. By the Baire property of f^{y_1}, there is a set $H \subset G^{y_1}$ of the second category with the Baire property such that $\operatorname{osc}_H f^{y_1} < \eta/4$. Let $I_2 \subset I_1$ be an open interval such that $I_2 \cap D_x \neq \emptyset$ and $f(x,v) < f(x,y_1) + \eta/4$ for each $x \in H$ and $v \in I_2 \cap D_x$. Then the set $B = (H \times I_2) \cap D$ is of the second category, has the Baire property and $\operatorname{osc}_B f < \eta$. Since $B \subset A_2 \subset A$, the proof is completed. □

Remark 0.8 *Observe that if a family of functions $f_t : \mathbb{R} \to \mathbb{R}, t \in S$, is such that for each $x \in \mathbb{R}$ and for each $\eta > 0$ there is an open set U such that x is not a density point of $\mathbb{R} \setminus U$ and if for each $t \in S$ and for each $u \in U$ we have $f_t(u) - f_t(x) < \eta$, then the functions $f_t, t \in S$, are strongly upper τ_d equiquasicontinuous.*

Remark 0.9 *Assume* **(MA)**. *There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ having all sections f_x τ_d continuous and τ_{ae} quasicontinuous and all sections f^y measurable such that for some τ_e continuous function $g : \mathbb{R} \to \mathbb{R}$ Carathéodory's superposition $h(x) = f(x, g(x)), x \in \mathbb{R}$, is not measurable.*

PROOF. Let C, $D \subset (0,1)$ be Cantor sets such that $C \subset D$, $\mu(C) = 0$ and every point $x \in C$ is a density point of the set D. There is a τ_e continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(D) \subset C$ and $g(x) \neq g(u)$ for $u \neq x$. Let $A \subset D$ be a nonmeasurable set and let $x_0, \ldots, x_\alpha, \ldots, \alpha < \omega$, be a transfinite sequence of all reals such that for each $\alpha < \omega$ the set $A_\alpha = \{x_\beta; \beta < \alpha\}$ is of measure zero. For each $x = x_\alpha \in A$ there is a τ_d continuous function $k_x : \mathbb{R} \to [0,1]$ such that $k_x(y) = 0$ if $y \in (\mathbb{R} \setminus D) \cup A_\alpha$, $y \neq g(x)$ and $k_x(g(x)) > 0$ ([1]). Similarly if $x = x_\alpha \in \mathbb{R} \setminus A$, then there is a τ_d continuous function $k_x : \mathbb{R} \to [0,1]$ such that $k_x(y) = 0$ if $y \in (\mathbb{R} \setminus D) \cup A_\alpha \cup \{g(x)\}$. Moreover let ϕ be a function having the same properties as the function f (constructed for the set D) from the proof of Remark 0.6. Let $f(x,y) = k_x(y)$ if $y \in D$ and $f(x,y) = \phi(y)$ if y is not in D. By Lemma 0.1 all sections f_x are τ_{ae} quasicontinuous and τ_d continuous. All sections $f^y, y \in D$, are equal to zero almost everywhere, and hence they are measurable. If $y \in \mathbb{R} \setminus D$, then the section f^y is constant. Moreover, $\{x; h(x) = f(x, g(x)) > 0\} = A$ and A is not measurable. So h is nonmeasurable. □

Remark 0.10 *There is a function $f : \mathbb{R}^2 \to [0,1]$ having all sections f_x and f^y strongly τ_d quasicontinuous such that the function $h(x) = f(x,x), x \in \mathbb{R}$, is nonmeasurable. For example, if $A \subset (0,1)$ is a nonmeasurable set, then the function $f(x,y) = 1$ for $y > f(x)$ or $y = f(x)$ whenever $x \in A$ and $f(x,y) = 0$ otherwise, satisfies the above condition.*

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