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T.S.S.R.K. Rao, Indian Statistical Institute,R.V. College Post, Bangalore – 560 059, INDIA, tss@isibang.ernet.in

ON A THEOREM OF DUNFORD, PETTIS AND PHILLIPS

Abstract

In this short note, we give a more 'direct' proof of a classical theorem of Dunford, Pettis and Phillips, that any weakly compact operator from $L^1(\mu)$ into a Banach space X is representable.

Introduction

Let $(\Lambda, \mathcal{A}, \mu)$ be a finite measure space and X a Banach space. Let us recall from [4], that an operator $T : L^1(\mu) \longrightarrow X$ is said to be representable if $\exists$ a $g \in L_\infty(\mu, X)$, such that

$$T(f) = \int_\Lambda fg d\mu \qquad \forall f \in L^1(\mu)$$

With this notation, the theorem of Dunford-Pettis-Phillips (Theorem 12, Page 75 of [4]) can be stated as :

Theorem *Every weakly compact operator from $L^1(\mu)$ into X is representable.*

We give below a more direct proof (than, we believe the one given in [4]) based on a recently uncovered 'Fact' due to de Reyna et al [3].This author recently gave a different proof of 'Fact' in [7] by using a result of Rosenthal that, seperability is euivalent to countable chain condition for weakly compact sets. A proof of this result of Rosenthal without using the Dunford and Pettis circle of ideas but only a result of Amir and Lindenstrauss that any weakly compact set is homeomorphic to a subset of $c_0(\Gamma)$(see[1]), is available in [2]. Hence there is no circularity involved in the arguments we will be presenting below.

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Fact *Let K be a compact Hausdorff space and μ is finite, regular Borel measure on K. Let $f : K \longrightarrow X$ be a function that is continuous when X has the weak topology, then f is Bochner μ-integrable.*

Proof of Theorem: Let K be the Stone space of $L^\infty(\mu)$ and $\wedge$ denote Gelfand isometry between $L^\infty(\mu)$ and C(K). Let $\hat{\mu}$ be the finite regular Borel measure defined on K via the relation

$$\widehat{(\mu)}(A) = \int \widehat{\chi}_A^{-1} d\mu$$

for any clopen set $A \subset K$. See [8].

This association, via simple functions, also extends to an isometry between $L^1(\mu)$ and $L^1(\hat{\mu})$. Griem remarks in [6], that since countably valued functions are uniformly dense in $L^\infty(\mu, X)$, the Gelfand map also extends as an isometry between the spaces $L^\infty(\mu, X)$ and $L^\infty(\hat{\mu}, X)$.

Hence there is no loss of generality in proving the theorem for $L^1(\hat{\mu})$ and X.

Let $T : L^1(\hat{\mu}) \longrightarrow X$ be a weakly compact operator. Since $L^1(\hat{\mu})^* = L^\infty(\hat{\mu}) = C(K), T^* : X^* \longrightarrow C(K)$ is weakly compact.

Let $\delta : K \longrightarrow C(K)^*$ be the Dirac map, which is a homeomorphism when $C(K)^*$ is equipped with the w^*-topology.

By Theorem 2, VI. 4.3 of [5], $T^{**} : C(K)^{**} \longrightarrow X$ is a w^*-weak continuous map.

Now $g = T^{**} \circ \delta : K \longrightarrow X$ is a weakly continuous function and by the 'Fact' quoted above, is $\hat{\mu}$-Bochner integrable. Hence $g \in L^\infty(\hat{\mu}, X)$. To verify the equation, $T(f) = \int_K fgd\hat{\mu}$, for $f \in L^1(\hat{\mu})$, it is enough to prove the equation when $f = \chi_A$ for a clopen set $A \subset K$.

For any $x^* \in X^*$

$$\begin{aligned} x^*(\int_A g(k)d\hat{\mu}(k)) &= \int_A x^*(T^{**}(\delta(k))d\hat{\mu}(k) \\ &= \int_A T^*(x^*)d\hat{\mu} = T^*(x^*)(\chi_A) \\ &= x^*(T(\chi_A)). \end{aligned}$$

Therefore $T(\chi_A) = \int_A gd\hat{\mu}$. □

Let $\mathcal{F}(L^1(\mu), X)$ denote the space of weakly compact operators and $WC(K, X)$ denote the space of X-valued functions on K that are continuous when X has the weak topology, equipped with the supremum norm.

Corollary *$\mathcal{F}(L^1(\mu), X)$ is isometric to the space $WC(K, X)$, where K is the Stone space of $L^\infty(\mu)$.*

Remark Some of the geometric properties of the space $WC(K, X)$, when K is any compact Hausdorff space, have been investigated in [3].

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