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AN INTEGRABILITY THEOREM FOR DIRICHLET SERIES

Abstract

In 1977 Leindler and Németh [4] proved a general theorem concerning integrability of a power series. The present note deals with a similar problem for Dirichlet series.

1 Let $\phi = \{\varphi \mid \varphi(u) \geq 0,\ \dfrac{\varphi(u)}{u}$ is non-decreasing and $\varphi(u)/u^{\delta} (\delta \geq 1)$ is non-increasing on $(0, \infty)\}$

$\Psi = \{\psi \mid$ inverse of ψ belongs to the class $\phi\}$

$P = \{\rho \mid \rho(u) \geq 0,$ is non-decreasing and $\rho(u^2) \leq R\rho(u), \quad u \in (0, \infty)\}$

$\bar{f}(x) =$ Inverse of $f(x)$.

With a view to generalize a theorem of Jain [2] and a previous result of Leindler [3], Leindler and Németh [4] established the following theorem concerning integrability of a power series.

Theorem 1 *Let $\mu(t)$ be a positive non-increasing function on the interval $(0, 1]$ such that*

$$\sum_{n=k}^{\infty} \mu(1/n) n^{-2} \leq M\mu(1/k)k^{-1} \tag{1.1}$$

and let $\{\alpha_n\}$ be a positive increasing sequence such that $\sum_1^{\infty} \dfrac{1}{n\alpha_n} < \infty$.

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Suppose $\rho \in P$, $\eta \in \phi$ or Ψ and $F(x) = \sum_0^\infty c_n x^n$, $0 \le x < 1$.
Then under the condition

$$c_n > -Kn^{-1}\bar{\eta}\left\{\frac{n}{\alpha_n \mu(1/n)\rho(n)}\right\}, \tag{1.2}$$

K is a positive constant,

$$\mu(1-x)\eta(|F(x)|)\rho(|F(x)|) \in L(0,1) \tag{1.3}$$

iff

$$\sum_{n=1}^{\infty} n^{-2}\mu(1/n)\rho(n)\eta\left(\sum_{k=0}^{n}|c_k|\right) < \infty. \tag{1.4}$$

In this note we propose to examine a similar problem for Dirichlet series. Let

$$f(t) = \sum_0^\infty c_k e^{-\lambda_k t}, \quad 0 < t \le \infty, \tag{1.5}$$

where $\lambda_0 = 0$, $1 = \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$. In what follows we assume that

(i) $\lambda_{nj} \sim \lambda_n \lambda_j$, $j = 1, 2, 3, \ldots, n \to \infty$,

(ii) $\{\lambda_{n+1} - \lambda_n\}$ is non-decreasing.

We write $W_n = \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}}$, $n \ge 1$.

Writing $\lambda_n = n$ and $x = e^{-t}$ the Dirichlet series reduces to the power series $\sum_0^\infty c_n x^n$.

The earliest known result concerning integrability of a Dirichlet series is due to Owen [6] which states:

$$\int_0^\infty f(t)^r t^{r/q-1} dt \le K\left[\sum_1^\infty c_n^p \lambda_n^{-(p+q-pq)/q}(\lambda_n - \lambda_{n-1})^{1-p}\right]^{r/p}, \tag{1.6}$$

where $c_n \ge 0$, $r \ge p \ge 1$, $q > 0$.

A generalization of (1.6) was later on given by Mulholland [5]. In what follows we prove the following general theorem for Dirichlet series which generalizes the result of Leindler and Németh when $\eta(x) = \varphi(x)$.

In order to state our theorem we need the concept of almost monotone functions [1][1]. A function f positive and finite on $I_b = [b, \infty)$ is said to be almost increasing on $I_c (c \geq b)$ if there exists a constant $M \geq 1$ such that $f(x) \leq Mf(y)$ for each $y \geq x \geq c$. Similarly f is said to be almost decreasing on I_c if there exists a constant m, $0 < m \leq 1$ such that $f(x) \geq mf(y)$ for each $y \geq x \geq c$. It is obvious that every increasing (decreasing) function is almost increasing (almost decreasing) but the converse, need not be true. It is known that a function f is almost increasing on I_c iff there exists an increasing function g on I_c such that $g(x) \asymp f(x)$ on I_c, that is to say $A \leq f(x)/g(x) \leq B$ for $x \in I_c$, where $0 < A < B < \infty$. A similar characterization is true for an almost decreasing function. For various other properties of almost monotone functions see [1].

We denote the class ϕ by ϕ^* when monotonic property is replaced by almost monotonicity, that is to say, we define

$$\begin{aligned}\phi^* = \{\varphi \mid \varphi(u) \geq 0,\ &\text{is increasing, } \frac{\varphi(u)}{u} \text{ is \textit{almost} non-decreasing and}\\ &\varphi(u)/u^\delta (\delta \geq 1) \text{ is non-increasing on } (0,\infty)\}, \text{ and}\end{aligned}$$

$$\begin{aligned}P^* = \{\rho \mid \rho(u) \geq 0,\ &\text{is \textit{almost} non-decreasing and}\\ &\rho(u^2) \leq R\rho(u), \quad u \in (0,\infty)\}.\end{aligned}$$

Theorem 2 *Let $\mu(t)$ be a positive almost non-increasing function on $(0,1]$ such that for $\rho \in P^*$*

$$\sum_{n=k}^{\infty} \mu(1/\lambda_n) W_{n+1} \rho(\lambda_n) \leq M\mu(1/\lambda_k)\lambda_{k+1}^{-1}\rho(\lambda_k) \tag{1.7}$$

and let $\{\alpha_n\}$ be a positive sequence such that

$$\sum_{1}^{\infty} \frac{\lambda_n W_{n+1}}{\alpha_n} < \infty. \tag{1.8}$$

Suppose $\varphi \in \phi^*$ and for $0 < t \leq \infty$ $f(t) = \sum_0^\infty c_n c^{-\lambda_n t}$, then under the condition

$$c_n > -K\frac{\lambda_{n+1} - \lambda_n}{\lambda_n}\bar{\varphi}\left\{\frac{\lambda_n}{\alpha_n \mu(1/\lambda_n)\rho(\lambda_n)}\right\} \tag{1.9}$$

$$\int_0^\infty e^{-t}\mu(1 - e^{-t})\varphi(|f(t)|)\rho(|f(t)|)dt < \infty \tag{1.10}$$

[1]This concept was introduced earlier by Mulholland [5] who termed it as quasi-monotone function.

iff

$$\sum_{1}^{\infty} W_{n+1}\mu(1/\lambda_n)\varphi\left(\sum_{k=0}^{n}|c_k|\right)\rho\left(\sum_{k=0}^{n}|c_k|\right)<\infty. \tag{1.11}$$

For $\lambda_n = n$, $e^{-t} = x$, $\eta(u) = \varphi(u)$ we deduce Theorem 1.

2 The proof of our theorem depends mainly on the following lemma.

Lemma 1 *Let $A(t) = \sum_0^\infty a_k e^{-\lambda_k t}$, $a_k \geq 0$, $0 < t \leq \infty$. If μ, φ and ρ satisfy the conditions of our theorem, then*

$$\int_0^\infty e^{-t}\mu(1-e^{-t})\varphi(A(t))\rho(A(t))dt < \infty \tag{2.1}$$

iff

$$\sum_{1}^{\infty} W_{n+1}\mu(1/\lambda_n)\varphi(S_n)\rho(S_n) < \infty, \tag{2.2}$$

where $S_n = \sum_{k=0}^n a_k$.

This includes, as a special case, a lemma of Leindler and Németh [4] for the case $\eta(u) = \varphi(u)$.

Proof. Writing $x = e^{-t}$ and using the convention that C denotes a positive constant not necessarily the same at each occurrence, we have

$$\begin{aligned}
&\int_0^\infty e^{-t}\mu(1-e^{-t})\varphi(A(t))\rho(A(t))dt \\
&= \int_0^1 \mu(1-x)\varphi\left(\sum_{k=0}^\infty a_k x^{\lambda_k}\right)\rho\left(\sum_{k=0}^\infty a_k x^{\lambda_k}\right)dx \\
&= \sum_{n=1}^\infty \int_{1-\frac{1}{\lambda_n}}^{1-\frac{1}{\lambda_{n+1}}} \mu(1-x)\varphi\left(\sum_{k=0}^\infty a_k x^{\lambda_k}\right)\rho\left(\sum_{0}^\infty a_k x^{\lambda_k}\right)dx \\
&> C^* \sum_{n=1}^\infty W_{n+1}\mu(1/\lambda_n)\varphi\left(\sum_{k=0}^n a_k\left(1-\frac{1}{\lambda_n}\right)^{\lambda_k}\right)\rho\left(\sum_{k=0}^n a_k\left(1-\frac{1}{\lambda_n}\right)^{\lambda_k}\right) \\
&\geq C\sum_{n=1}^\infty W_{n+1}\mu(1/\lambda_n)\varphi\left(\sum_{k=0}^n a_k\left(1-\frac{1}{\lambda_n}\right)^{\lambda_n}\right)\rho\left(\sum_{k=0}^n a_k\left(1-\frac{1}{\lambda_n}\right)^{\lambda_n}\right) \\
&\geq C\sum_{n=2}^\infty W_{n+1}\mu(1/\lambda_n)\varphi\left(\sum_{k=0}^n a_k\left(1-\frac{1}{\lambda_2}\right)^{\lambda_2}\right)\rho\left(\sum_{k=0}^n a_k\left(1-\frac{1}{\lambda_2}\right)^{\lambda_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq C\sum_{n=2}^{\infty} W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\varphi\left(\left(1-\frac{1}{\lambda_2}\right)^{\lambda_2} S_n\right)\rho\left(\left(1-\frac{1}{\lambda_2}\right)^{\lambda_2} S_n\right)\\
&\geq C\sum_{n=2}^{\infty} W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\varphi(S_n)\rho(S_n),
\end{aligned}
$$

since $\left(1-\dfrac{1}{x}\right)^x$ is increasing for $x \geq 1$ and $\varphi(cx) \geq K\varphi(x)$, $0 < c < 1$ and $\rho(ax) \leq K\rho(x)$, $a > 0$, $\forall\, x > d > 0$, $\rho(d) \neq 0$.

Thus (2.1) $\Rightarrow$ (2.2). Conversely,

(2.3)

$$
\begin{aligned}
I &= \int_0^{\infty} e^{-t}\mu(1-e^{-t})\varphi(A(t))\rho(A(t))dt\\
&\leq C\sum_{n=2}^{\infty}\mu\left(\frac{1}{\lambda_n}\right)W_n\varphi\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{\lambda_n}\right)^{\lambda_k}\right)\rho\left(\sum_{k=0}^{\infty} a_k\left(1-\frac{1}{\lambda_n}\right)^{\lambda_k}\right)\\
&\leq C\sum_{n=2}^{\infty}\mu\left(\frac{1}{\lambda_n}\right)W_n\varphi\left(\sum_{j=0}^{\infty}\sum_{k=nj}^{(j+1)n} a_k\left(1-\frac{1}{\lambda_n}\right)^{\lambda_k}\right)\\
&\quad\times\rho\left(\sum_{j=0}^{\infty}\sum_{k=nj}^{(j+1)n} a_k\left(1-\frac{1}{\lambda_n}\right)^{\lambda_k}\right)\\
&< C\sum_{n=2}^{\infty} W_n\mu(1/\lambda_n)\varphi\left(\sum_{j=0}^{\infty}\sum_{k=nj}^{(j+1)n}\left(1-\frac{1}{\lambda_n}\right)^{\lambda_{nj}} a_k\right)\\
&\quad\times\rho\left(\sum_{j=0}^{\infty}\sum_{k=nj}^{(j+1)n}\left(1-\frac{1}{\lambda_n}\right)^{\lambda_{nj}} a_k\right).
\end{aligned}
$$

Now,

$$\sum_{j=0}^{\infty}\sum_{k=nj}^{(j+1)n}(1-\frac{1}{\lambda_n})^{\lambda_{nj}}a_k$$
$$=\sum_{j=0}^{\infty}(1-\frac{1}{\lambda_n})^{\lambda_{nj}}\sum_{k=nj}^{(j+1)n}a_k$$
$$\leq\sum_{j=0}^{\infty}e^{-\lambda_j}\sum_{k=nj}^{(j+1)n}a_k<\sum_{j=0}^{\infty}e^{-\lambda_j}\sum_{k=0}^{(j+1)n}a_k$$
$$=\sum_{j=0}^{\infty}e^{-\lambda_j}S_{(j+1)n}=\sum_{j=1}^{\infty}e^{-\lambda_{j-1}}S_{nj}.$$

Hence from (2.3)

$$I\leq C\sum_{n=2}^{\infty}\mu(1/\lambda_n)W_n\varphi\left[\sum_{i=1}^{\infty}e^{-\lambda_{i-1}}S_{ni}\right]\rho\left[\sum_{i=1}^{\infty}e^{-\lambda_{i-1}}S_{ni}\right].\tag{2.4}$$

Using the inequality (see [5], p.490)

$$\varphi\left\{\frac{\sum_{i=1}^{\infty}a_ib_i}{\sum_{i=1}^{\infty}a_i}\right\}\rho\left\{\frac{\sum_{i=1}^{\infty}a_ib_i}{\sum_{i=1}^{\infty}a_i}\right\}\leq K\frac{\sum_{i=1}^{\infty}a_i\varphi(b_i)\rho(b_i)}{\sum_{i=1}^{\infty}a_i}\tag{2.5}$$

and in view of the fact that

$$\lambda_{m+1}=\sum_{k=0}^{m}(\lambda_{k+1}-\lambda_k)\geq(\lambda_1-\lambda_0)(m+1)=m+1$$

implies

$$\sum_{i=1}^{\infty}e^{-\lambda_{i-1}}\leq\sum_{i=1}^{\infty}e^{-(i-1)}=\frac{e}{e-1},$$

we have on writing $\sum_{i=1}^{\infty}e^{-\lambda_{i-1}}=L>1$

$$\varphi\left[\sum_{i=1}^{\infty}e^{-\lambda_{i-1}}LS_{ni}/L\right]\rho\left[\sum_{i=1}^{\infty}e^{-\lambda_{i-1}}LS_{ni}/L\right]$$
$$\leq C\sum_{i=1}^{\infty}e^{-\lambda_{i-1}}\varphi(LS_{ni})\rho(LS_{ni})$$
$$\leq C\sum_{i=1}^{\infty}e^{-\lambda_{i-1}}\varphi(S_{ni})\rho(S_{ni}),$$

since $\rho(Lx) \leq K\rho(x)$ for $x \geq c > 0$. Hence from (2.4)

$$I \leq C \sum_{n=2}^{\infty} W_n \mu(1/\lambda_n) \sum_{i=1}^{\infty} e^{-\lambda_{i-1}} \varphi(S_{ni}) \rho(S_{ni}).$$

Since

$$W_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1}} \leq \frac{\lambda_{ni} - \lambda_{ni-1}}{\lambda_n \lambda_{n-1}} = W_{ni} \frac{\lambda_{ni} \lambda_{ni-1}}{\lambda_n \lambda_{n-1}}$$
$$\leq C W_{ni} \lambda_i^2 \frac{\lambda_n}{\lambda_{n-1}} \leq C W_{ni} \lambda_i^2$$

and in view of μ being almost non-increasing function

$$\mu\left(\frac{1}{\lambda_n}\right) \leq \frac{1}{m} \mu(1/\lambda_{ni}).$$

Thus

$$I \leq C \sum_{i=1}^{\infty} e^{-\lambda_{i-1}} \lambda_i^2 \sum_{n=2}^{\infty} W_{ni} \mu\left(\frac{1}{\lambda_{ni}}\right) \varphi(S_{ni}) \rho(S_{ni})$$
$$\leq C \sum_{i=1}^{\infty} e^{-\lambda_{i-1}} \lambda_i^2 \sum_{m=1}^{\infty} \mu\left(\frac{1}{\lambda_m}\right) W_{m+1} \varphi(S_m) \rho(S_m)$$
$$\leq C \sum_{n=1}^{\infty} W_{n+1} \mu(1/\lambda_n) \varphi(S_n) \rho(S_n),$$

since

$$\sum_{i=1}^{\infty} \lambda_i^2 e^{-\lambda_{i-1}} \leq 1 + C \sum_{i=2}^{\infty} \lambda_{i-1}^2 e^{-\lambda_{i-1}}$$
$$\leq 1 + C \sum_{i=2}^{\infty} \frac{\lambda_{i-1}^2}{\lambda_{i-1}^4} \leq 1 + C \sum_{i=2}^{\infty} \frac{1}{\lambda_{i-1}^2}$$
$$\leq 1 + C \sum_{i=2}^{\infty} \frac{1}{(i-1)^2} < \infty.$$

This completes the proof our lemma. □

3 Proof of the theorem. Let

$$F(t) = \sum_{k=0}^{\infty} a_k e^{-\lambda_k t}, \; 0 < t \leq \infty,$$

with

$$a_k = K\left(\frac{\lambda_{k+1}-\lambda_k}{\lambda_k}\right)\bar{\varphi}\left\{\frac{\lambda_k}{\alpha_k\mu\left(\frac{1}{\lambda_k}\right)\rho(\lambda_k)}\right\}.$$

Using the inequality:

$$\sum_{n=1}^{\infty} X_n\varphi(S_n) \le K\sum_{n=1}^{\infty} X_n\varphi\left(\frac{a_n}{X_n}\sum_{k=n}^{\infty} X_k\right)$$

which holds for any $X_n > 0$ and $a_n \ge 0$ (See [4] p. 100) we have on writing $X_n = \mu\left(\dfrac{1}{\lambda_n}\right)W_{n+1}\rho(\lambda_n)$

$$\begin{aligned}
&\sum_1^{\infty}\mu\left(\frac{1}{\lambda_n}\right)W_{n+1}\varphi(S_n)\rho(\lambda_n)\\
&\le K\sum_{n=1}^{\infty}\mu\left(\frac{1}{\lambda_n}\right)W_{n+1}\rho(\lambda_n)\varphi\left\{\frac{a_n\sum_{k=n}^{\infty}\mu\left(\frac{1}{\lambda_k}\right)W_{k+1}\rho(\lambda_k)}{\mu(1/\lambda_n)W_{n+1}\rho(\lambda_n)}\right\}\\
&\le K\sum_{n=1}^{\infty}\mu\left(\frac{1}{\lambda_n}\right)W_{n+1}\rho(\lambda_n)\varphi\left\{\frac{Ka_n\mu\left(\frac{1}{\lambda_n}\right)\lambda_{n+1}^{-1}\rho(\lambda_n)}{\mu\left(\frac{1}{\lambda_n}\right)W_{n+1}\rho(\lambda_n)}\right\}\\
&= K\sum_{n=1}^{\infty}W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\rho(\lambda_n)\varphi\left\{\frac{a_n}{W_{n+1}\lambda_{n+1}}\right\}
\end{aligned}$$

$$\begin{aligned}
&\le K\sum_{n=1}^{\infty}W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\rho(\lambda_n)\frac{\lambda_n}{\alpha_n\mu(1/\lambda_n)\rho(\lambda_n)}\\
&= K\sum_{n=1}^{\infty}\frac{W_{n+1}\lambda_n}{\alpha_n} < \infty
\end{aligned}$$

in view of (1.7) and (1.8).

This implies that $S_n = 0\left(\lambda_n^2\right)$. Hence in view of the property of ρ we have

$$\sum_1^{\infty} W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\varphi(S_n)\rho(S_n) < \infty. \tag{3.1}$$

Hence by our lemma

$$\int_0^{\infty} e^{-t}\mu(1-e^{-t})\varphi(F(t))\rho(F(t))dt < \infty. \tag{3.2}$$

From (1.9) $a_n + c_n > 0$ for all sufficiently large n, hence $f(t) + F(t) = \sum_0^\infty (a_k + c_k)e^{-\lambda_k t}$ implies in view of our lemma that

$$\int_0^\infty e^{-t}\mu(1-e^{-t})\varphi(|F(t)+f(t)|)\rho(|F(t)+f(t)|)dt < \infty \tag{3.3}$$

iff

$$\sum_{n=1}^\infty W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\varphi\left(\sum_{k=0}^n (a_k + c_k)\right)\rho\left(\sum_0^n (a_k + c_k)\right) < \infty. \tag{3.4}$$

From (2.5) it follows that

$$\varphi(a+b)\rho(a+b) \le K\{\varphi(a)\rho(a) + \varphi(b)\rho(b)\}, \ a > 0, \ b > 0. \tag{3.5}$$

Suppose

$$\int_0^\infty e^{-t}\mu(1-e^{-t})\varphi(|f(t)|)\rho(|f(t)|)dt < \infty$$

then in view of (3.2) and (3.5), (3.3) is true, which in turn, implies (3.4). Now

$$|c_n| \le 2a_n + c_n$$

implies that $\sum_{k=0}^n |c_k| \le \sum_{k=0}^n (c_k + a_k) + \sum_{k=0}^n a_k$ so that

$$\begin{aligned}
&\sum_{n=1}^\infty W_{n+1}\mu(1/\lambda_n)\varphi\left(\sum_{k=0}^n |c_k|\right)\rho\left(\sum_{k=0}^n |c_k|\right) \\
&\quad \le \sum_{n=1}^\infty W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\varphi\left(\sum_{k=0}^n (a_k + c_k) + \sum_{k=0}^n a_k\right)\rho\left(\sum_{k=0}^n (c_k + a_k) + \sum_{k=0}^n a_k\right) \\
&\quad \le K\sum_{n=1}^\infty W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\varphi\left(\sum_{k=0}^n (a_k + c_k)\right)\rho\left(\sum_{k=0}^n (a_k + c_k)\right) \\
&\qquad + K\sum_{n=1}^\infty W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\varphi(S_n)\rho(S_n) < \infty
\end{aligned}$$

by virtue of (3.1) and (3.4). Conversely, suppose

$$\sum_{n=1}^\infty W_{n+1}\mu\left(\frac{1}{\lambda_n}\right)\varphi\left(\sum_{k=0}^n |c_k|\right)\rho\left(\sum_{k=0}^n |c_k|\right) < \infty.$$

Then in view of (3.1) and (3.5), (3.4) is true. Using (3.2) and (3.3) we have in view of (3.5)

$$\int_0^\infty e^{-t}\mu(1-e^{-t})\varphi(|f(t)|)\rho(|f(t)|)dt$$
$$\leq \int_0^\infty e^{-t}\mu(1-e^{-t})\varphi(|f(t)+F(t)|+F(t))\rho(|f(t)+F(t)|+F(t))dt < \infty.$$

This completes the proof of our Theorem.

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