Real Analysis Exchange
Vol. 20(2), 1994/5, pp. 681–698

Pradipta Bandyopadhyay, Stat–Math Division, Indian Statistical Institute, 203, B. T. Road, Calcutta 700 035, INDIA, e-mail: pradipta@@isical.ernet.in

A. K. Roy, Stat–Math Division, Indian Statistical Institute, 203, B. T. Road, Calcutta 700 035, INDIA, e-mail: ashoke@@isical.ernet.in

EXTREME CONTRACTIONS IN $\mathcal{L}(\ell_2^p, \ell_2^q)$ AND THE MAZUR INTERSECTION PROPERTY IN $\ell_2^p \otimes_\pi \ell_2^q$

Abstract

In this paper, we show that the projective tensor product of a two-dimensional ℓ^p space with a two-dimensional ℓ^q space never has the Mazur Intersection Property for a large range of values of p and q. For this purpose, we characterize the extreme contractions from ℓ_2^p to ℓ_2^q and obtain their closure.

1 Introduction

A Banach space is said to have the Mazur Intersection Property (MIP) if every closed bounded convex set is the intersection of closed balls. In a finite-dimensional space X, this is equivalent to the extreme points of the dual unit ball $B(X^*)$ being norm dense in the dual sphere $S(X^*)$. And, in general,

Theorem 1.1 *For a Banach space X, the following are equivalent :*

(*a*) *The w^*-denting points of $B(X^*)$ are norm dense in $S(X^*)$.*

(*b*) *X has the MIP.*

(*c*) *Every support mapping on X maps norm dense subsets of $S(X)$ to norm dense subsets of $S(X^*)$.*

Key Words: Mazur Intersection Property, Extreme Contractions, ℓ^p spaces, Projective Tensor Product.
Mathematical Reviews subject classification: 46B20
Received by the editors November 3, 1994

(see [2] or [4] for details and related results)

Using this characterization, Ruess and Stegall [15] have shown that the injective tensor product of two Banach spaces of dimension ≥ 2 never has the MIP. And Sersouri [17] has shown that in fact there is a two-dimensional compact convex set in $X \otimes_\epsilon Y$ that is not an intersection of balls.

The situation appears to be much more difficult for projective tensor product spaces, since the extremal structure of the unit ball of the dual of $X \otimes_\pi Y$, i.e., $\mathcal{L}(X, Y^*)$ (see e.g., [3, Chapter VIII]), is known only in some very special cases and no pattern is discernible even in these cases for a reasonable conjecture to be made in general. See [9] or [12] for a survey.

The simplest situation arises in a Hilbert space, where the extreme contractions are characterized as isometries and coisometries, by Kadison [11] in the complex case (see also [10]) and by Grzaslewicz [6] in the real case. And it immediately follows that the projective tensor product of two Hilbert spaces never has the MIP.

Complications already increase significantly if we move on to ℓ^p-spaces. In fact, the complete picture eludes us even for two-dimensional ℓ^p-spaces. Here we show that the projective tensor product of a two-dimensional ℓ^p space with a two-dimensional ℓ^q space never has the MIP for a large range of values of p and q. For this purpose, we characterize the extreme contractions from ℓ_2^p to ℓ_2^q and obtain their closure. Some of the results about extreme contractions were proved earlier in [5, 6, 8] through different techniques. Our approach is similar to that of [12] for the case $p = q$ with complex scalars. We, however, work only with real scalars. This technique also lends itself naturally to the computation of the closure.

A major portion of this work is contained in the first-named author's Ph. D. Thesis [1] written under the supervision of the second author. We take this opportunity to thank a referee whose detailed comments on the paper led to considerable improvement of its exposition.

2 Extreme Contractions in $\mathcal{L}(\ell_2^p, \ell_2^q)$

NOTATIONS : For $1 < p < \infty$, $x = (x_1, x_2) \in \ell_2^p$ with $\|x\| = 1$, define $x^{p-1} = (sgn(x_1)|x_1|^{p-1}, sgn(x_2)|x_2|^{p-1})$ and $x^o = (-x_2, x_1)$. Notice that, in general, x^{p-1} is the unique norming functional of x and $\{x, (x^o)^{p-1}\}$ is a basis for ℓ_2^p, and if $p = 2$, $x^{p-1} = x$ and $\{x, x^o\}$ is orthonormal. Denote the vectors (1, 0) and (0, 1) by e_1 and e_2 respectively.

We will need the following inequality [14, Lemma 1.e.14]

Lemma 2.1 *Let $1 < p \leq q < \infty$. Let $\alpha = \pm\sqrt{(p-1)/(q-1)}$. Then*

$$\left[\frac{1}{2}\{|1+\alpha r|^q + |1-\alpha r|^q\}\right]^{1/q} \leq \left[\frac{1}{2}\{|1+r|^p + |1-r|^p\}\right]^{1/p}$$

for all $r \in \mathbb{R}$ with strict inequality holding for $r \neq 0$.

Theorem 2.1 *For $1 < p,\ q < \infty$, an operator $T : \ell_2^p \longrightarrow \ell_2^q$ with $\|T\| = 1$ is an extreme contraction*

(*i*) [6] *for $p = q = 2$, if and only if T is an isometry.*

(*ii*) [8] *for $p = 2 \neq q$, if and only if T satisfies one of the following*

(*a*) *T attains its norm on two linearly independent vectors.*

(*b*) *T is of the form*

$$T = \begin{cases} \boldsymbol{x} \otimes e_i & \text{if} \quad q < 2 \\ \boldsymbol{x} \otimes \boldsymbol{y} + s\boldsymbol{x}^o \otimes (\boldsymbol{y}^o)^{q-1} & \text{if} \quad q > 2 \end{cases}$$

where $\boldsymbol{x}$ is any unit vector and, in the second case, $|y_i|^q = \frac{1}{2}$ and $s = \pm\frac{1}{\sqrt{(q-1)}}2^{(q-2)/q}$.

(*iii*) [8] *for $p \neq 2 = q$, if and only if T satisfies one of the following*

(*a*) *T attains its norm on two linearly independent vectors.*

(*b*) *T is of the form*

$$T = \begin{cases} e_i \otimes \boldsymbol{y} & \text{if} \quad p > 2 \\ \boldsymbol{x}^{p-1} \otimes \boldsymbol{y} + s\boldsymbol{x}^o \otimes \boldsymbol{y}^o & \text{if} \quad p < 2 \end{cases}$$

where $\boldsymbol{y}$ is any unit vector and, in the second case, $|x_i|^p = \frac{1}{2}$ and $s = \pm\sqrt{(p-1)}2^{(2-p)/p}$.

(*iv*) [5] *for $p = q \neq 2$, if and only if T satisfies one of the following*

(*a*) *T attains its norm on two linearly independent vectors.*

(*b*) *T is of the form*

$$T = \begin{cases} e_i \otimes \boldsymbol{y} & \text{if} \quad p > 2, \quad y_1 y_2 \neq 0 \\ \boldsymbol{x}^{p-1} \otimes e_j & \text{if} \quad p < 2, \quad x_1 x_2 \neq 0 \end{cases}$$

(*v*) *for $1 < q < 2 < p < \infty$, if and only if T satisfies one of the following*

(*a*) *T attains its norm on two linearly independent vectors.*

(*b*) *$T = \boldsymbol{x}^{p-1} \otimes \boldsymbol{y}$ with $\boldsymbol{x}$, $\boldsymbol{y}$ unit vectors and $x_1 x_2 y_1 y_2 = 0$.*

PROOF. Let $T : \ell_2^p \longrightarrow \ell_2^q$, $||T|| = 1$. Then there exists $\boldsymbol{x} = (x_1, x_2) \in \ell_2^p$ such that $||\boldsymbol{x}|| = 1 = ||T\boldsymbol{x}||$. Let

$$T\boldsymbol{x} = \boldsymbol{y} = (y_1, y_2) \text{ and } I_{xy} = \{T : ||T|| \le 1, T\boldsymbol{x} = \boldsymbol{y}\}.$$

Then for any $T \in I_{xy}$, $(T - \boldsymbol{x}^{p-1} \otimes \boldsymbol{y})$ annihilates $\boldsymbol{x}$ and so is of rank ≤ 1, whence $(T - \boldsymbol{x}^{p-1} \otimes \boldsymbol{y}) = \boldsymbol{x}^o \otimes \boldsymbol{u}$, for some $\boldsymbol{u} \in \ell_2^p$. Further, $T^*(\boldsymbol{y}^{q-1}) = \boldsymbol{x}^{p-1}$, that is $(T^* - \boldsymbol{y} \otimes \boldsymbol{x}^{p-1})$ annihilates $\boldsymbol{y}^{q-1}$, whence $(T^* - \boldsymbol{y} \otimes \boldsymbol{x}^{p-1}) = (\boldsymbol{y}^o)^{q-1} \otimes \boldsymbol{v}$, for some $\boldsymbol{v} \in \ell_2^q$. Combining, T must be of the form

$$T_s = \boldsymbol{x}^{p-1} \otimes \boldsymbol{y} + s\boldsymbol{x}^o \otimes (\boldsymbol{y}^o)^{q-1}, \text{ for some } s \in \mathbb{R}$$

In other words, $I_{xy} = \{T_s : s \in \mathbb{R}, ||T_s|| \le 1\}$. That is, I_{xy} is a line segment (could be degenerate) in the unit ball, and, its end points are extreme.

As in [12], pre- or post-multiplying by diag$(sgn(x_1), sgn(x_2))$, diag$(sgn(y_1), sgn(y_2))$ and permutation matrices, if necessary — each of which is an isometry — we may assume $x_1 \ge x_2 \ge 0$, $y_1 \ge y_2 \ge 0$.

For $r \in \mathbb{R}$, denote by $f_p(\boldsymbol{x}, r) = \boldsymbol{x} + r(\boldsymbol{x}^o)^{p-1}$ and $F_p(\boldsymbol{x}, r) = ||f_p(\boldsymbol{x}, r)||^p$.

Then $F_p(\boldsymbol{x}, r) = |x_1 - rx_2^{p-1}|^p + |x_2 + rx_1^{p-1}|^p$. Clearly, if $p = 2$ or $x_2 = 0$, $F_p(\boldsymbol{x}, r) = 1 + |r|^p$. Otherwise,

$$\begin{aligned} F_p(\boldsymbol{x}, r) &= x_2^{-p} \cdot |rx_2^p - x_1x_2|^p + x_1^{-p} \cdot |rx_1^p + x_1x_2|^p \\ &= x_2^{-p} G(rx_2^p - x_1x_2) + x_1^{-p} G(rx_1^p + x_1x_2) \end{aligned}$$

where $G(u) = |u|^p$. Thus

$$\frac{\partial}{\partial r} F_p(\boldsymbol{x}, r) = G'(rx_2^p - x_1x_2) + G'(rx_1^p + x_1x_2) \tag{1}$$

and $G'(0) = 0$, $G'(u) = p \cdot sgn(u) \cdot |u|^{p-1}$. Clearly, (1) also holds for $p = 2$ and $x_2 = 0$.

Now, $G'(u)$ is an odd function, positive and strictly increasing for $u > 0$. Since the two arguments of G' in (1) add up to r, we have if $r > 0$ (resp. $r < 0$), the one larger in absolute value is positive (resp. negative), and so, $\frac{\partial}{\partial r} F_p(\boldsymbol{x}, r)$ is positive (resp. negative), i.e., $F_p(\boldsymbol{x}, r)$ is strictly increasing (resp. decreasing) in $r > 0$ (resp. $r < 0$).

Further, if $p \ne 2$ and $x_2 \ne 0$

$$\left.\begin{aligned} F_p(\boldsymbol{x}, 0) &= 1, \quad F_p'(\boldsymbol{x}, 0) = 0, \\ F_p''(\boldsymbol{x}, 0) &= p(p-1)(x_1x_2)^{p-2}, \\ F_p'''(\boldsymbol{x}, 0) &= p(p-1)(p-2)(x_1x_2)^{p-3}[x_1^p - x_2^p], \\ F_p''''(\boldsymbol{x}, 0) &= p(p-1)(p-2)(p-3)(x_1x_2)^{p-4}[1 - 3(x_1x_2)^p] \end{aligned}\right\} \tag{2}$$

and so, if $1 < q < \infty$, for $H_{pq}(x, r) = [F_p(x, r)]^{q/p}$, we have,

$$\left.\begin{array}{rcl} H_{pq}(x,0) & = & 1, \quad H'_{pq}(x,0) \;=\; 0, \\ H''_{pq}(x,0) & = & q(p-1)(x_1x_2)^{p-2}, \\ H'''_{pq}(x,0) & = & q(p-1)(p-2)(x_1x_2)^{p-3}[x_1^p - x_2^p], \\ H''''_{pq}(x,0) & = & q(p-1)(x_1x_2)^{p-4}[(p-2)(p-3) \\ & & \quad - 3(x_1x_2)^p\{(p-2)(p-3)+(p-q)(p-1)\}] \end{array}\right\} \tag{3}$$

where the derivatives are taken with respect to r.

Now, $T_s(f_p(x, r)) = f_q(y, rs)$, and thus for any $r \neq 0$, $\|T_s(f_p(x, r))\|^q = F_q(y, rs)$ is strictly increasing in $s \geq 0$ and strictly decreasing in $s \leq 0$, and $F_q(y, rs)$ is unbounded in s. Now, if $r \neq 0$,

$$F_q(y, 0) = 1 = [F_p(x, 0)]^{q/p} < [F_p(x, r)]^{q/p}$$

So, there exists unique $s_+(x, y, r) > 0$ and unique $s_-(x, y, r) < 0$ such that

$$F_q(y, rs_\pm) = [F_p(x, r)]^{q/p} \tag{4}$$

And the quantity on the LHS becomes smaller or larger than the one in the RHS according as $|s|$ gets smaller or larger. Evidently, such $s_\pm$ also exist for $(x^o)^{p-1}$, which we denote by $f_p(x, \infty)$. In fact, in this case, $|s_\pm(x, y, \infty)| = \|(x^o)^{p-1}\|/\|(y^o)^{q-1}\|$. Notice that $s_\pm$ is a continuous function of $r \neq 0$ and elementary examples show that $\lim_{r \to 0} s_\pm(x, y, r)$ may not even exist. Let

$$s_+^*(x, y) = \inf\{s_+(x, y, r) : r \neq 0\} \quad s_-^*(x, y) = \sup\{s_-(x, y, r) : r \neq 0\}$$

Clearly, $T_s \in I_{xy}$ if and only if $s_-^* \leq s \leq s_+^*$, i.e., $T_{s_\pm^*}$ are end points of I_{xy} and hence are extreme. Also let

$$\begin{array}{rcl} s_+^{**}(x, y) & = & \liminf\limits_{r \to 0} s_+(x, y, r) \;\stackrel{\text{def}}{=}\; \sup\limits_{\varepsilon > 0} \inf\{s_+(x, y, r) : |r| < \varepsilon\} \\ s_-^{**}(x, y) & = & \limsup\limits_{r \to 0} s_-(x, y, r) \;\stackrel{\text{def}}{=}\; \inf\limits_{\varepsilon > 0} \sup\{s_-(x, y, r) : |r| < \varepsilon\} \end{array}$$

Note that if we put $J_{xy} = \{s : T_s \text{ is contractive in a neighbourhood of } x\}$, then $s_-^{**} = \inf J_{xy}$ and $s_+^{**} = \sup J_{xy}$, though $s_\pm^{**}$ may not necessarily belong to J_{xy}. Clearly, $s_-^{**} \leq s_-^* \leq 0 \leq s_+^* \leq s_+^{**}$.

Now, either $s_\pm^*$ equals $s_\pm(x, y, r)$ for some $r \neq 0$ (including $r = \infty$), in which case $s_\pm^* \neq 0$ and $T_{s_\pm^*}$ attain their norm on two linearly independent vectors, or $s_\pm^* = s_\pm^{**}$.

Note that T attains its norm on two linearly independent vectors if and only if T^* attains its norm on two linearly independent vectors. Moreover, any such T is exposed, and hence strongly exposed.

Thus to complete the proof, the task that remains is to identify all (if any) extreme contractions that attain their norm in exactly one direction (called 'of the desired type' in the sequel). Then $s_{\pm}^{*} = s_{\pm}^{**}$, in which case (*i*) $|s_{\pm}^{**}| < \infty$, (*ii*) $s_{\pm}^{**} \in J_{xy}$, in fact, (*iii*) $T_{s_{\pm}^{**}} \in I_{xy}$.

Therefore, in different cases, we proceed to successively check these three conditions and whenever we reach a contradiction, we conclude that $s_{\pm}^{*} \neq s_{\pm}^{**}$ and $T_{s_{\pm}^{*}}$ is not of the desired type. And in case all the three conditions are satisfied, we check whether it attains its norm in any direction other than that of x and only if it does not, we get an extreme contraction of the desired type. This line of reasoning is exemplified in the analysis of cases (II) and (IV) below. However, in case (I), we can directly calculate $s_{\pm}^{*}$.

CASE(I) : (*i*) $p = 2$ and either $q = 2$ or $y_2 = 0$; (*ii*) $q = 2$ and either $p = 2$ or $x_2 = 0$; (*iii*) $p \neq 2 \neq q$ and $x_2 = 0 = y_2$.

$$\begin{aligned} T_s \text{ is a contraction} \quad &\Longleftrightarrow \quad F_q(y, rs) \leq [F_p(x,r)]^{q/p} \quad \text{for all } r \\ &\Longleftrightarrow \quad 1 + |rs|^q \leq [1 + |r|^p]^{q/p} \quad \text{for all } r \\ &\Longleftrightarrow \quad |s|^q \leq \frac{[1+|r|^p]^{q/p} - 1}{|r|^q} \quad \text{for all } r \neq 0 \end{aligned}$$

Note that the RHS $\equiv 1$ if $p = q$ and is strictly decreasing (resp. increasing) in $|r|$ for $q > p$ (resp. $q < p$).

So, if $p = q$, $s_{\pm} \equiv \pm 1$, and hence, $s_{\pm}^{*} = s_{\pm}^{**} = \pm 1$ and $T_{s_{\pm}^{*}}$ are isometries. And, if $p \neq q$, the infimum of the RHS over $r \neq 0$ yields

$$|s_{\pm}^{*}|^q = \begin{cases} 1 & \text{if} \quad q > p \\ 0 & \text{if} \quad q < p \end{cases}$$

So, if $q < p$, $s_{\pm}^{*} = 0$, and T_0 is an extreme contraction of the desired type. And if $q > p$, $s_{\pm}^{*} = s_{\pm}(\infty) = \pm 1$ with $T_{\pm 1}$ attaining its norm at both x and $(x^{o})^{p-1}$. It is interesting to note that if $p \neq 2 \neq q$, T_1 in this case is the identity operator.

For the remaining cases, we calculate $s_{\pm}^{**}$. Let $\{r_n\}$ be a sequence of real numbers such that $r_n \longrightarrow 0$ and $s_{\pm}(r_n) \longrightarrow s_{\pm}^{**}$. If we assume $|s_{\pm}^{**}| < \infty$, then $\{s_{\pm}(r_n)\}$ is a bounded sequence. Now, by (4),

$$F_q(y, r_n s_{\pm}(r_n)) = [F_p(x, r_n)]^{q/p} \tag{5}$$

CASE (II) : $q \neq 2$, $y_2 > 0$ and either $p = 2$ or $x_2 = 0$.

In this case, subtracting 1 from both side of (5), dividing by r_n^2 and taking limit as $n \longrightarrow \infty$, we get by L'Hospital's rule and (2) that

$$\text{LHS} \longrightarrow \frac{1}{2} q(q-1) s^2 (y_1 y_2)^{q-2},$$

where $s = s_\pm^{**}$ and

$$\text{RHS} \longrightarrow \begin{cases} 0 & \text{if} \quad p > 2 \\ \infty & \text{if} \quad p < 2 \\ \dfrac{q}{2} & \text{if} \quad p = 2 \end{cases}$$

So, if $p < 2$, we have a contradiction, whence $T_{s_\pm^*}$ is not of the desired type, and if $p > 2$, $s_\pm^* = s_\pm^{**} = 0$, and T_0 is extreme, and clearly of the desired type.

If $p = 2$, we have

$$s^2(q-1)(y_1y_2)^{q-2} = 1 \tag{6}$$

Now, if the s given by (6) belongs to J_{xy}, we must have

$$F_q(\boldsymbol{y}, rs) \leq [F_p(\boldsymbol{x}, r)]^{q/p} \qquad \text{for all small } r \neq 0 \tag{7}$$

Comparing the Taylor expansion of the two sides around $r = 0$ (for the LHS use (2)), we see that the coefficients of 1, r and r^2 on both sides are equal, whence the inequality (7) for small r implies the corresponding inequality for the coefficient of r^3 on both sides, which, for $r > 0$ and $r < 0$, leads to the equality

$$\frac{1}{6}s^3q(q-1)(q-2)(y_1y_2)^{q-3}(y_1^q - y_2^q) = 0 \tag{8}$$

Combining equations (6) and (8), we have $y_1^q = y_2^q = 1/2$, $s^2 = \frac{1}{(q-1)}4^{(q-2)/q}$. But again the equality in (8) pushes the inequality down to the coefficients of r^4, i.e.,

$$\begin{aligned} \frac{1}{8}q(q-2) &\geq \frac{1}{24}s^4q(q-1)(q-2)(q-3)(y_1y_2)^{q-4}[1 - 3(y_1y_2)^q] \\ \text{or} \quad 3(q-2) &\geq \frac{(q-2)(q-3)}{(q-1)} \end{aligned}$$

Now for $q < 2$, this leads to a contradiction, so that $T_{s_\pm^*}$ is not of the desired type. On the other hand, by Lemma 2.1 for $p = 2$ and $q > 2$, we have that T_s with the above parameters is a contraction that attains its norm only in the direction of $\boldsymbol{x}$ and hence, is of the desired type.

CASE (III) : $p \neq 2$, $x_2 > 0$ and either $q = 2$ or $y_2 = 0$.

This situation is dual to case (II) above.

CASE(IV) : $p \neq 2 \neq q$ and $x_2 > 0$, $y_2 > 0$.

In this case too, subtracting 1 from both side of (5), dividing by r_n^2 and taking limit as $n \longrightarrow \infty$, we get by (2) and (3)

$$(q-1)(y_1y_2)^{q-2}s^2 = (p-1)(x_1x_2)^{p-2} \tag{9}$$

where $s = s_{\pm}^{**}$.

So, if $s \in J_{xy}$, comparing the Taylor expansion of the two sides of (7) around $r = 0$ (use (2) for the LHS and (3) for the RHS), by arguments similar to Case (II) ($p = 2$), we must have

$$s^3(q-1)(q-2)(y_1y_2)^{q-3}(y_1^q - y_2^q) = (p-1)(p-2)(x_1x_2)^{p-3}(x_1^p - x_2^p) \tag{10}$$

and

$$\begin{aligned} s^4(q-1)(q-2)(q-3)(y_1y_2)^{q-4}[1-3(y_1y_2)^q] \le (p-1)(x_1x_2)^{p-4} \cdot \\ [(p-2)(p-3)\{1-3(x_1x_2)^p\} - 3(p-q)(p-1)(x_1x_2)^p] \end{aligned} \tag{11}$$

Eliminating s from (9) and (10) and using the fact that x, y are unit vectors, we get

$$\frac{(q-2)^2}{(q-1)}\left[\frac{1}{(y_1y_2)^q} - 4\right] = \frac{(p-2)^2}{(p-1)}\left[\frac{1}{(x_1x_2)^p} - 4\right] \tag{12}$$

Also, dividing (11) by the square of (9), we get

$$\frac{(q-2)(q-3)}{(q-1)}\left[\frac{1}{(y_1y_2)^q} - 3\right] \le \frac{(p-2)(p-3)}{(p-1)}\left[\frac{1}{(x_1x_2)^p} - 3\right] - 3(p-q) \tag{13}$$

Notice that for $p = q$, we get from (12) that $x_i = y_i$, $i = 1, 2$, whence from (9), $s = \pm 1$, and from (10), $y_1 = y_2 = x_1 = x_2$ for $s = -1$. Thus,

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which, clearly, are isometries.

Now, let $p \neq q$. From (12) and (13), writing $\dfrac{1}{(x_1x_2)^p} = A$, we get

$$\frac{(p-2)^2(q-3)}{(q-2)(p-1)}(A-4) + \frac{(q-2)(q-3)}{(q-1)} \le \frac{(p-2)(p-3)}{(p-1)}(A-3) - 3(p-q)$$

Simplifying we get

$$\frac{(q-p)(p-2)}{(p-1)(q-2)} \cdot A \le \frac{2q(q-p)(pq-p-q)}{(p-1)(q-1)(q-2)}$$

So, if (a) $1 < q < 2 < p < \infty$, or, (b) $1 < p < q < 2$, or, (c) $2 < p < q < \infty$, we have

$$A \le \frac{2q(pq-p-q)}{(p-2)(q-1)} \qquad \text{i.e.,} \qquad A - 4 \le \frac{2(q-2)(pq-p-q+2)}{(p-2)(q-1)} \tag{14}$$

And if (d) $1 < p < 2 < q < \infty$, or, (e) $1 < q < p < 2$, or, (f) $2 < q < p < \infty$, we have

$$A \geq \frac{2q(pq-p-q)}{(p-2)(q-1)} \qquad \text{i.e.,} \qquad A-4 \geq \frac{2(q-2)(pq-p-q+2)}{(p-2)(q-1)} \tag{15}$$

Now, $(x_1x_2)^p = 1/A$ and $x_1^p + x_2^p = 1$, so $0 < 1/A \leq 1/4$, i.e., $A \geq 4$ or $A - 4 \geq 0$.

But since $(pq-p-q+2) = (p-1)(q-1)+1$ is always positive for $1 < p$, $q < \infty$, if $1 < q < 2 < p < \infty$, i.e., in case (a) above, we reach a contradiction at this point, whence $T_{s_\pm^*}$ is not of the desired type. □

REMARK. (*a*) For $p > q \geq 2$ and $y_2 = 0$, the same result, as in Case (III) above, has been obtained by Kan [13, Lemma 6.2] for complex scalars too.

(*b*) Recently we have come to know that P. Scherwentke [16] has proved a special case of Theorem 2.1 (v), i.e., when $p > 2$ and $1/p + 1/q = 1$, using techniques similar to [8].

3 Partial Results in Remaining Cases

In the last part of the proof of Theorem 2.1, the conditions (b) and (e) are dual to (c) and (f) respectively. And in the cases (b) and (c), the inequality (14) implies

$$\frac{1}{2} \leq x_1^p \leq \frac{1}{2}\left[1 + \sqrt{\frac{(q-2)(pq-p-q+2)}{q(pq-p-q)}}\right] \tag{16}$$

while in cases (e) and (f), the inequality (15) implies

$$\frac{1}{2}\left[1 + \sqrt{\frac{(q-2)(pq-p-q+2)}{q(pq-p-q)}}\right] \leq x_1^p < 1 \tag{17}$$

Now, in cases (b), (c), (e) and (f), we have from (10) that for $s < 0$, both sides of (10) must be 0, i.e., we must have $x_1^p = x_2^p = 1/2 = y_1^q = y_2^q$. But then in cases (e) and (f), we have a contradiction. So, in these two cases, s_-^* gives extreme contractions not of the desired type.

Also in case (d), (15) is always satisfied and (10) implies that for $s > 0$, $x_1^p = x_2^p = 1/2 = y_1^q = y_2^q$.

Now, from (9) it follows that $T_{s_\pm^{**}}$ is a contraction if and only if

$$\begin{aligned} &\left[y_1^q \cdot \left|1 + \alpha(y_2/y_1)^{q/2} r\right|^q + y_2^q \cdot \left|1 - \alpha(y_1/y_2)^{q/2} r\right|^q\right]^{1/q} \\ \leq\ &\left[x_1^p \cdot \left|1 + (x_2/x_1)^{p/2} r\right|^p + x_2^p \cdot \left|1 - (x_1/x_2)^{p/2} r\right|^p\right]^{1/p} \end{aligned} \tag{18}$$

for all $r \in I\!R$, where $\alpha = \pm\sqrt{(p-1)/(q-1)}$ with the sign being that of $s_{\pm}^{**}$. Thus for the particular case of $x_1^p = x_2^p = 1/2 = y_1^q = y_2^q$, we have by Lemma 2.1 that in cases (b), (c) and (d) for both $s > 0$ and $s < 0$, we get extreme contractions of the desired type.

Thus, *modulo* duality, we are left with the following cases unsolved :

(1) Case (b) with $x_1^p > 1/2$ satisfying (16) for $s > 0$ with y_1 given by (12).

(2) Case (e) with x_1 satisfying (17) for $s > 0$ with y_1 given by (12).

(3) Case (d) with $x_1^p > 1/2$ and $s < 0$ with y_1 given by (12).

In the remaining part of this section, we prove that in case (b), i.e., for $1 < p < q < 2$, for $1/2 < x_1^p \leq 1/q$, we get extreme contractions of the desired type. Specifically, we prove

Lemma 3.1 *Let* $1 < p < q < 2$, $1/2 < x_1^p \leq 1/q$, *then* (18) *holds for all* $r \in I\!R$, *with equality only for* $r = 0$.

PROOF. For notational simplicity, put $x_i^p = a_i$, $y_i^q = b_i$, $i = 1, 2$ and $a_2/a_1 = u$, $b_2/b_1 = v$. Notice that, in this notation, (12) becomes

$$(2-q)\alpha v^{-1/2}(1-v) \quad = \quad (2-p)u^{-1/2}(1-u) \tag{19}$$

which implies $0 < v \leq u \leq 1$. It is also not difficult to see that

$$\alpha v^{-1/2} \leq u^{-1/2} \iff a_1 \leq \frac{1}{q} \tag{20}$$

Also, in our notation (18) becomes

$$\begin{aligned} &\left[\frac{1}{1+v}\cdot\left|1+\alpha v^{1/2}r\right|^q + \frac{v}{1+v}\cdot\left|1-\alpha v^{-1/2}r\right|^q\right]^{1/q} \\ \leq\ &\left[\frac{1}{1+u}\cdot\left|1+u^{1/2}r\right|^p + \frac{u}{1+u}\cdot\left|1-u^{-1/2}r\right|^p\right]^{1/p} \end{aligned} \tag{21}$$

CASE I : $0 \leq r \leq u^{1/2}$

Expanding both LHS^q and RHS^p by Binomial series, and noting that $\alpha^2 = (p-1)/(q-1)$ and $(1+x)^{q/p} \geq 1 + \frac{q}{p}x$ for all $x \geq 0$, it suffices to show that

$$\begin{aligned} 0 \ &\leq\ (2-q)\cdots(k-1-q)\alpha^{k-2}v^{-(k-2)/2}\left[\frac{1+(-1)^{k-2}v^{k-1}}{1+v}\right] \\ &\leq\ (2-p)\cdots(k-1-p)u^{-(k-2)/2}\left[\frac{1+(-1)^{k-2}u^{k-1}}{1+u}\right] \end{aligned} \tag{22}$$

for all $k \geq 3$.

Now, as $0 < v < u < 1$, both $1 + (-1)^{k-2}u^{k-1}$ and $1 + (-1)^{k-2}v^{k-1}$ are nonnegative, i.e., the first inequality in (22) follows. Also for $k = 3$, the second inequality in (22) is an equality by (19). And for $k \geq 4$, dividing both sides of (22) by that of (19), it suffices to show

$$\begin{aligned} & (3-q)\cdots(k-1-q)\alpha^{k-3}v^{-(k-3)/2}\left[\frac{1+(-1)^{k-2}v^{k-1}}{1-v^2}\right] \\ \leq \ & (3-p)\cdots(k-1-p)u^{-(k-3)/2}\left[\frac{1+(-1)^{k-2}u^{k-1}}{1-u^2}\right] \end{aligned}$$

But for $k \geq 4$, $(3-q)\cdots(k-1-q) < (3-p)\cdots(k-1-p)$, and by (20), $\alpha^{k-3}v^{-(k-3)/2} \leq u^{-(k-3)/2}$. Also, it is not difficult to see that for any $k \geq 4$, $[1+(-1)^{k-2}x^{k-1}]/(1-x^2)$ is strictly increasing for $0 < x < 1$. Since $0 < v < u < 1$, Case I follows.

Notice that for $r = u^{1/2}$, we get

$$\left[\frac{1}{1+v}\cdot\left(1+\alpha v^{1/2}u^{1/2}\right)^q + \frac{v}{1+v}\cdot\left(1-\alpha v^{-1/2}u^{1/2}\right)^q\right] \leq \left[(1+u)^{p-1}\right]^{q/p} \tag{23}$$

CASE II : $r \leq 0$ and $r \geq u^{1/2}$.

Notice that if we put $t = -u^{-1/2}r/(1-u^{-1/2}r)$, (21) becomes

$$\begin{aligned} & \left[\frac{1}{1+v}\cdot|1-(1+c)t|^q + \frac{v}{1+v}\cdot|1-(1-c/v)t|^q\right]^{1/q} \\ \leq \ & \left[\frac{1}{1+u}\cdot|1-(1+u)t|^p + \frac{u}{1+u}\right]^{1/p} \end{aligned} \tag{24}$$

where $c = \alpha u^{1/2}v^{1/2}$, and the ranges $r \leq 0$ and $r \geq u^{1/2}$ become $0 \leq t < 1$ and $t > 1$ respectively. Thus, we have to prove (24) for $t \geq 0$, $t \neq 1$.

Notice that by (20), $c \leq v$ and $c \leq \alpha u < u$. Put

$$\begin{aligned} \phi_1(t) &= \frac{1}{1+v}\left[|1-(1+c)t|^q + v\cdot|1-(1-c/v)t|^q\right] \text{ and} \\ \phi_2(t) &= \frac{1}{1+u}\left[|1-(1+u)t|^p + u\right] \end{aligned}$$

Put $f(t) = q\log\phi_2(t) - p\log\phi_1(t)$. We have to show $f(t) > 0$ for $t \neq 0$. Now, $f'(t) = (q\phi_1(t)\phi_2'(t) - p\phi_1'(t)\phi_2(t))/(\phi_1(t)\phi_2(t))$, so that $f'(t) >, =,$ or $<$ 0 according as $q\phi_1(t)\phi_2'(t) - p\phi_1'(t)\phi_2(t) >, =,$ or < 0; or, equivalently,

$$\begin{aligned} & sgn[1-(1-c/v)t]|1-(1-c/v)t|^{q-1}\cdot[1-(uv+c)\cdot g(t)] \\ <, =, \text{ or } > \ & sgn[1-(1+c)t]\cdot|1-(1+c)t|^{q-1}\cdot[1+(u-c)\cdot g(t)] \end{aligned} \tag{25}$$

where $g(t) = \{1 - sgn[1-(1+u)t] \cdot |1-(1+u)t|^{p-1}\}/c(1+u)$.

Notice that $g'(t) = c^{-1}(p-1)|1-(1+u)t|^{p-2} \geq 0$, and hence, $g(t)$ is strictly increasing with $g(0) = 0$.

SUBCASE 1 : $0 \leq t \leq 1/(1+c)$.

Since $f(0) = 0$, and it suffices to prove $f'(t) \geq 0$, or, in (25), LHS $\leq$ RHS. Notice that in this case, every factor on the two sides of (25), except possibly the third term on the LHS, is nonnegative. And the third term on the LHS is decreasing, positive at $t = 0$ and is ≤ 0 at $t = 1/(1+u)$. When this term is ≤ 0, we have nothing to prove. And thus it suffices to prove

$$\left[\frac{1-(1-c/v)t}{1-(1+c)t}\right]^{q-1} \leq \frac{[1+(u-c)\cdot g(t)]}{[1-(uv+c)\cdot g(t)]}$$

for the values of t for which $g(t) < 1/(uv+c)$, which exclude the values $1/(1+u) \leq t \leq 1/(1+c)$.

Again since in this range all the factors are positive and the two sides are equal at $t = 0$, taking logarithm and differentiating, it suffices to show

$$\frac{(q-1)c}{v[1-(1-c/v)t]\cdot[1-(1+c)t]} \leq \frac{ug'(t)}{[1+(u-c)\cdot g(t)]\cdot[1-(uv+c)\cdot g(t)]}$$

Simplifying the expressions and putting $s = (1+u)t$, this is equivalent to

$$A\cdot[\frac{c^2}{(1-s)} + v(1-s)^{p-1}] + D\cdot[c^2(1-s) + u^2v(1-s)^{1-p}] + B\cdot[c^2 - uv] \geq 0$$

where

$$\begin{aligned} A &= (uv+c)(u-c) > 0 \\ B &= (uv+c)(1+c) + (v-c)(u-c) \\ D &= (1+c)(v-c) \geq 0 \end{aligned}$$

Now, in the range $0 \leq s < 1$ and so, we can expand the LHS by Binomial and geometric series. Note that

$$A + D + B = v(1+u)^2 \text{ and } A + u^2D - uB = -c^2(1+u)^2$$

whence the constant term on the LHS is

$$c^2(A+D+B) + v(A+u^2D-uB) = 0$$

On the other hand, we have from (19), that

$$\begin{aligned} A - D &= (1+u)[c(1-v) - v(1-u)] = \frac{c(1+u)(1-v)(q-p)}{2-p} \\ A - u^2D &= c(1+u)[u(1-v) - c(1-u)] = uv(1-u^2)\frac{q-p}{(q-1)(2-q)} > 0 \end{aligned}$$

whence the coefficient of s on the LHS is

$$c^2(A-D)-(p-1)v(A-u^2D)=c^2(1+u)(q-p)\left[\frac{c(1-v)}{2-p}-\frac{v(1-u)}{2-q}\right]=0$$

Therefore, and since $1<p<2$, we have the coefficient of s^k, $k\geq 2$, is

$$\begin{aligned}
&A\left\{c^2-\frac{v(p-1)(2-p)\cdots(k-p)}{k!}\right\}+Du^2v\frac{(p-1)p(p+1)\cdots(p+k-2)}{k!}\\
&\geq Ac^2-\frac{(p-1)(2-p)\cdots(k-p)}{k!}\cdot v(A-u^2D)\\
&=Dc^2+v(A-u^2D)(p-1)\left[1-\frac{(2-p)\cdots(k-p)}{k!}\right]>0
\end{aligned}$$

SUBCASE 2 : $1/(1+c)<t<v/(v-c)$.
Since $(|x|+|y|)^a\geq |x|^a+|y|^a$ for $a>1$, we have that in (24)

$$\begin{aligned}
\mathrm{RHS}^q &\geq \left(\frac{1}{1+u}\right)^{q/p}[(1+u)t-1]^q+\left(\frac{u}{1+u}\right)^{q/p}\\
\mathrm{LHS}^q &= \frac{1}{1+v}\cdot[(1+c)t-1]^q+\frac{v}{1+v}\cdot[1-(1-c/v)t]^q
\end{aligned}$$

Comparing the first term of the two sides, it suffices to show

$$\begin{aligned}
\frac{1}{1+v}\cdot[(1+c)t-1]^q &\leq \left(\frac{1}{1+u}\right)^{q/p}[(1+u)t-1]^q\\
\text{or, }\left[\frac{(1+c)t-1}{(1+u)t-1}\right] &\leq \frac{(1+v)^{1/q}}{(1+u)^{1/p}}
\end{aligned}$$

for $t>1/(1+c)$, the LHS is increasing, and the maximum value at "$t=\infty$" is $(1+c)/(1+u)$. Thus it suffices to show that

$$(1+c)\leq(1+v)^{1/q}\cdot(1+u)^{1-1/p}$$

but this follows from (23).

And comparing the second term, we need to show

$$\left(\frac{u}{1+u}\right)^{q/p}\geq\frac{v}{1+v}\cdot[1-(1-c/v)t]^q$$

For $1/(1+c)<t<v/(v-c)$, the RHS is decreasing and it suffices to prove

$$\left(\frac{u}{1+u}\right)^{q/p}\geq\frac{v}{1+v}\cdot\left[1-\frac{(1-c/v)}{(1+c)}\right]^q=\frac{v}{1+v}\cdot\left[\frac{(c+c/v)}{(1+c)}\right]^q$$

Now, if we consider the function

$$h(x) = \frac{1}{1+v} \cdot \left[\frac{x-c}{1+x}\right]^q + \frac{v}{1+v} \cdot \left[\frac{x+c/v}{1+x}\right]^q$$

we see that

$$h'(x) = q \cdot \frac{(1+c)(x-c)^{q-1} + (v-c)(x+c/v)^{q-1}}{(1+v)(1+x)^{q+1}}$$

whence $h(x)$ is increasing for $x \geq c$, and the inequality (24) at $t = 1/(1+u)$, yields

$$\left(\frac{u}{1+u}\right)^{q/p} \geq h(u) \geq h(c)$$

This proves the subcase 2.

SUBCASE 3 : $t \geq v/(v-c)$. Notice that if $v = c$, this case does not arise.

In this case, the LHS of (25) is ≥ 0, while the RHS < 0, whence $f'(t) < 0$, and the minimum value of f is attained for "$t = \infty$". Now that this value is > 0 follows from (23). This completes the proof of Case II, and hence, of the Lemma. □

In the particular case $a_1 = 1/q$, replacing $r/\sqrt{q-1}$ by t, we get the following interesting inequality, the case $q = 2$ being immediate from Lemma 2.1.

Corollary 3.1 *Let* $1 < p < q \leq 2$. *Then*

$$\left[\frac{1}{p}|1+(p-1)t|^q + \frac{p-1}{p}|1-t|^q\right]^{1/q} \leq \left[\frac{1}{q}|1+(q-1)t|^p + \frac{q-1}{q}|1-t|^p\right]^{1/p}$$

for all $t \in \mathbb{R}$ *with strict inequality holding for* $t \neq 0$.

REMARK. We do not know whether the range $1/2 \leq x_1^p \leq 1/q$ exhausts all values of x_1^p for which we get a contraction. However, it is not very difficult to see that we cannot have the entire range in (16). Indeed, when x_1^p is the right end point, we do not even get a contractive T, as in that case tracing our arguments back we find that the coefficients of r^4 in the Taylor expansion of the two sides of (7) must be equal, and hence, as in the case of r^3, we must have equality of the coefficients of r^5 as well. But then direct computations reveal a contradiction.

4 The Closure of Extreme Contractions

We now obtain the closure of the extreme contractions in the cases described in Theorem 2.1.

Theorem 4.1 *In all the cases described in Theorem 2.1, except the case $p = q \neq 2$, the set of extreme contractions is closed.*

And in the case $p = q \neq 2$, the closure of extreme contractions may have operators of the form diag$(1, s)$, $|s| < 1$ (upto isometric factors of signum or permutation matrices) as additional elements.

PROOF. In case (i), the result is obvious. And case (iii) is dual to case (ii). Now, in the cases (ii) and (v), the set of operators of the form (b) is clearly closed. And in case (iv), the closure of the set of operators of the form (b) contains only the operators $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, i, $j = 1, 2$ in addition.

Let us consider the set of operators of the type (a) in cases (ii), (iv) and (v). Let $\{T_n\}$ be a sequence of operators of the type (a). Let $T_n \longrightarrow T$ in operator norm. Let $\boldsymbol{x}_n = (x_{n1}, x_{n2})$ be such that $||\boldsymbol{x}_n|| = 1 = ||T_n \boldsymbol{x}_n||$. Let $T_n \boldsymbol{x}_n = \boldsymbol{y}_n = (y_{n1}, y_{n2})$. Then T_n is of the form

$$T_n = \boldsymbol{x}_n^{p-1} \otimes \boldsymbol{y}_n + s_{\pm}^{*}(\boldsymbol{x}_n, \boldsymbol{y}_n)\boldsymbol{x}_n^{o} \otimes (\boldsymbol{y}_n^{o})^{q-1}$$

where $s_{\pm}^{*}(\boldsymbol{x}_n, \boldsymbol{y}_n)$ is as in our earlier discussion. For notational simplicity, write $s_{\pm}^{*}(\boldsymbol{x}_n, \boldsymbol{y}_n) = \pm s_n$. Passing to a subsequence, if necessary, assume $\boldsymbol{x}_n \longrightarrow \boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y}_n \longrightarrow \boldsymbol{y} = (y_1, y_2)$ (by compactness of the unit balls of ℓ_2^p and ℓ_2^q), and all the s_n's have the same sign, without loss of generality, positive.

Clearly, $||T|| = 1$ and $T\boldsymbol{x} = \boldsymbol{y}$, whence T is of the form

$$T = \boldsymbol{x}^{p-1} \otimes \boldsymbol{y} + s\boldsymbol{x}^{o} \otimes (\boldsymbol{y}^{o})^{q-1}$$

Also, as $T_n \longrightarrow T$,

$$s_n = ||T_n - \boldsymbol{x}_n^{p-1} \otimes \boldsymbol{y}_n|| / ||\boldsymbol{x}_n^{o}|| \cdot ||(\boldsymbol{y}_n^{o})^{q-1}|| \longrightarrow ||T - \boldsymbol{x}^{p-1} \otimes \boldsymbol{y}|| / ||\boldsymbol{x}^{o}|| \cdot ||(\boldsymbol{y}^{o})^{q-1}||,$$

i.e., $\{s_n\}$ is convergent. Clearly, $s_n \longrightarrow s$.

Now, since T_n is of the type (a), there exists $\boldsymbol{z}_n = \boldsymbol{x}_n + r_n(\boldsymbol{x}_n^{o})^{p-1}$ with $r_n \neq 0 \in \mathbb{R}$ such that $||T_n \boldsymbol{z}_n|| = ||\boldsymbol{z}_n||$. Again we may assume all r_n's are of the same sign, in particular positive and $r_n \longrightarrow r$, where $0 \leq r \leq \infty$. If $0 < r < \infty$, $\boldsymbol{z}_n \longrightarrow \boldsymbol{z} = \boldsymbol{x} + r(\boldsymbol{x}^{o})^{p-1}$ and $||T\boldsymbol{z}|| = ||\boldsymbol{z}||$, i.e., T is also of the type (a). Also, if $r_n \longrightarrow \infty$, let $\boldsymbol{u}_n = \boldsymbol{z}_n / ||\boldsymbol{z}_n||$. Then $\boldsymbol{u}_n \longrightarrow \boldsymbol{u} = (\boldsymbol{x}^{o})^{p-1} / ||(\boldsymbol{x}^{o})^{p-1}||$ and $||T\boldsymbol{u}|| = 1$, so that T again is of the type (a).

Now, suppose $r_n \longrightarrow 0$. Then from $||T_n \boldsymbol{z}_n|| = ||\boldsymbol{z}_n||$ we have

$$F_q(\boldsymbol{y}_n, r_n s_n) - [F_p(\boldsymbol{x}_n, r_n)]^{q/p} = 0 \tag{26}$$

For (ii), if $p = 2$ and $q < 2$, since T_n is of type (a), we have $y_{n1} y_{n2} \neq 0$ for all n. And if $q > 2$, we have two possibilities; either there is a subsequence for which $y_{n1} y_{n2} = 0$, or, eventually $y_{n1} y_{n2} \neq 0$. In the first case, we restrict ourselves only to that subsequence, and we have, by case (I), $s_n = 1$ for all n, whence $s = 1$. Also, $y_1 y_2 = 0$. So, $T = \boldsymbol{x} \otimes \boldsymbol{e}_i + \boldsymbol{x}^o \otimes \boldsymbol{e}_j$ $(i \neq j)$, and it is clear that T is of type (a) (see case (I)). And in the second case, we assume $y_{n1} y_{n2} \neq 0$ for all n. Then dividing (26) by r_n^2 and taking limit as $n \longrightarrow \infty$, we get by L'Hospital's rule

$$(q-1)s^2(y_1 y_2)^{q-2} - 1 = 0 \tag{27}$$

If $q > 2$, for $y_1 y_2 = 0$, this leads to a contradiction, whence $y_1 y_2 \neq 0$. Then (6) and (27) coincides, i.e., we have $s = s_+^{**}(\boldsymbol{x}, \boldsymbol{y})$. Now, our analysis in case (II) shows that only for $y_1^q = y_2^q = 1/2$, s_+^{**} gives a contraction (which is an extreme contraction of type (b)). And in every other case, we run into a contradiction, i.e., we must have $r_n \not\to 0$.

And if $q < 2$, for $y_1 y_2 = 0$, (27) makes sense only if $s = 0$. In that case, $T = \boldsymbol{x} \otimes \boldsymbol{e}_i$, which, by case (I), is an extreme contraction of the type (b). And for $y_1 y_2 \neq 0$, we again have $s = s_+^{**}(\boldsymbol{x}, \boldsymbol{y})$ and our analysis in case (II) shows that this case always leads to a contradiction.

So, in both the cases, the closure of the set of operators of the type (a) contains at most operators of type (b), and therefore, the set of extreme contractions is closed.

In (*iv*), i.e., if $p = q$, by duality, it suffices to consider $p > 2$. Since T_n is of type (a), we have three possibilities; (1) either there is a subsequence for which both $x_{n1} x_{n2} = 0$ and $y_{n1} y_{n2} = 0$, or, (2) there is a subsequence for which $x_{n1} x_{n2} \neq 0$ and $y_{n1} y_{n2} = 0$, or, (3) eventually both $x_{n1} x_{n2} \neq 0$ and $y_{n1} y_{n2} \neq 0$.

In the first case, we again restrict ourselves only to that subsequence, and we have, by case (I), $s_n = 1$ for all n, whence $s = 1$. Also, $x_1 x_2 = 0$ and $y_1 y_2 = 0$. Now again by case (I), T is of type (a).

In cases (2) and (3), dividing (26) by r_n^2 and taking limit — through a subsequence if necessary — as $n \longrightarrow \infty$, we get in the second case

$$(x_1 x_2)^{p-2} = 0$$

and in the third case

$$(y_1 y_2)^{p-2} s^2 = (x_1 x_2)^{p-2}$$

So, in case (2), $y_1 y_2 = 0$, and we have a contradiction unless $x_1 x_2 = 0$. And in that case, T is of the form $\operatorname{diag}(1, s)$ upto isometric factors of signum or permutation matrices. Now, T is a contraction for $-1 \leq s \leq 1$ and is extreme (in fact, an isometry) only for $s = \pm 1$. However, we do not know precisely if they actually belong to the closure.

In case (3), if $x_1 x_2 = 0$, we get a contradiction unless $y_1 y_2 = 0$ or $s = 0$. If $y_1 y_2 \neq 0$, $s = 0 = s_{\pm}^*(\boldsymbol{x}, \boldsymbol{y})$, whence T is extreme. And if $y_1 y_2 = 0$, we get the conclusions as in case (2). If $x_1x_2 \neq 0$, $y_1y_2 = 0$ leads to a contradiction, and if $y_1y_2 \neq 0$, $s = s_+^{**}(\boldsymbol{x}, \boldsymbol{y})$, so that T is an isometry and hence is of type (a).

In case (v), i.e., if $1 < q < 2 < p < \infty$, since T_n is of type (a), we must have $x_{n1}x_{n2}y_{n1}y_{n2} \neq 0$ for all n. And a similar argument leads to

$$s^2(q-1)(y_1y_2)^{q-2} = (p-1)(x_1x_2)^{p-2}$$

If $x_1x_2 \neq 0$ the only situation that does not lead to any contradiction — either immediate or to the fact that T is a contraction — is both $y_1y_2 = 0$ and $s = 0$. And in that case, $s = 0 = s_\pm^*(\boldsymbol{x}, \boldsymbol{y})$, so that T is extreme. And if $x_1x_2 = 0$, we must have $s = 0$, in which case, by cases (I) and (II), $s = 0 = s_\pm^*(\boldsymbol{x}, \boldsymbol{y})$ and T is extreme. Thus in this case too, the set of extreme contractions is closed. □

Theorem 4.2 *In each of the following cases of* $1 < p,\ q < \infty$, $\ell_2^p \otimes_\pi \ell_2^q$ *lacks the MIP :*

(i) *p and q are conjugate exponents, i.e.,* $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) *Either p or q is equal to* 2.

(iii) $2 < p,\ q < \infty$.

PROOF. The dual of $\ell_2^p \otimes_\pi \ell_2^q$ is $\mathcal{L}(\ell_2^p, \ell_2^{q'})$, where $\frac{1}{q} + \frac{1}{q'} = 1$ and the closure of extreme contractions in none of the above cases contains norm 1 operators of the form $\boldsymbol{x}^{p-1} \otimes \boldsymbol{y}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are unit vectors with $x_1x_2y_1y_2 \neq 0$. □

REMARK. The fact that operators of the above form do not belong to the closure of extreme contractions in any of these cases seems to suggest that this is a general phenomenon. It is possible that this is happens in higher dimensions as well. Can one give a proof of this without precisely characterizing the extreme contractions? What seems to be required is a more tractable necessary condition for extremality, or, for belonging to the closure of extreme contractions.

References

[1] Bandyopadhyay P., *The Mazur Intersection Property in Banach Spaces and Related Topics*, Ph. D. Thesis, Indian Statistical Institute, Calcutta (February, 1991).

[2] P. Bandyopadhyay, *The Mazur Intersection Property for Families of Closed Bounded Convex Sets in Banach Spaces*, Colloq. Math. **LXIII** (1992), 45–56.

[3] Diestel J. and Uhl J. J., Jr., *Vector Measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, R. I. (1977).

[4] J. R. Giles, D. A. Gregory and B. Sims, *Characterization of Normed Linear Spaces with the Mazur's Intersection Property*, Bull. Austral. Math. Soc. **18** (1978), 105–123.

[5] Grzaslewicz R., *Extreme Operators on 2-dimensional ℓ^p-spaces*, Colloq. Math. **44** (1981), 309–315.

[6] Grzaslewicz R., *Extreme Contractions on Real Hilbert Spaces*, Math. Ann. **261** (1982), 463–466.

[7] Grzaslewicz R., *Exposed Points of the Unit Ball of $\mathcal{L}(H)$*, Math. Z. **193** (1986), 595–596.

[8] Grzaslewicz R., *Extremal Structure of $\mathcal{L}(\ell^2_m, \ell^p_n)$*, Linear and Multilinear Algebra **24** (1989), 117–125.

[9] Grzaslewicz R., *Survey of Main Results about Extreme Operators on Classical Banach Spaces*, (preprint).

[10] Halmos P. R., *A Hilbert Space Problem Book*, New Jersey (1967).

[11] Kadison R. V., *Isometries of Operator Algebras*, Ann. Math. **54** (1951), 325–338.

[12] Kan C. H., *A Class of Extreme L_p Contractions, $p \neq 1, 2, \infty$, and Real 2×2 Extreme Matrices*, Illinois J. Math., **30** (1986), 612–635.

[13] Kan C. H., *Norming Vectors of Linear Operators between L_p Spaces*, Pacific J. Math., **150** (1991), 309–327.

[14] Lindenstrauss J. and Tzafriri L., *Classical Banach Spaces*, Vol. II, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **97**, Springer Verlag (1979).

[15] Ruess W. M. and Stegall C. P., *Weak*-Denting Points in Duals of Operator Spaces*, Proceedings of the Missouri Conference on Banach Spaces, Springer Lecture Notes **1166** (1985), 158–168.

[16] Scherwentke P., *On the Geometry of $\mathcal{L}(\ell^p_2, \ell^q_2)$ and $\ell^q_2 \otimes_\varepsilon \ell^q_2$*, Arch. Math., (to appear).

[17] Sersouri A., *Smoothness in Spaces of Compact Operators*, Bull. Austral. Math. Soc. **38** (1988), 221–225.