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SOLUTION OF THE BAIRE ORDER PROBLEM OF MAULDIN

Let X be an uncountable separable and complete metric space (briefly called *Polish*) and let I be a σ-ideal of subsets of X such that $X \notin I$. Denote by C_I the family of all functions $f : X \to \mathbb{R}$ whose sets of points of discontinuity are in I. Then put $B_0(I) = C_I$ and for each ordinal $\alpha > 0$ define $B_\alpha(I)$ as the family of all pointwise limits of sequences of functions from $\bigcup_{\gamma<\alpha} B_\gamma(I)$.

It is easy to check that the *Baire system* $B_\alpha(I), \alpha \leq \omega_1$, has the following properties:

- $B_{\omega_1}(I)$ is closed under pointwise limits, i.e. $B_{\omega_1+1}(I) = B_{\omega_1}(I)$,
- for $I = \{\emptyset\}$ we have the classical Baire system (denoted by $B_\alpha, \alpha \leq \omega_1$).

Now we define $r(I) = min\{\alpha \leq \omega_1 : B_{\alpha+1}(I) = B_\alpha(I)\}$ which is called the *Baire order* of C_I.

In [M2] Mauldin posed the following problem:
If $0 < \alpha < \omega_1$, is there a σ-ideal I_α of the first category subsets of $[0, 1]$, which contains all sets of Lebesgue measure 0 such that the family of all functions which are continuous except for a set in this σ-ideal I_α has Baire order α ?

Note that in the above question, σ-ideals are required to contain all singletons $\{x\}$; a σ-ideal which has that property is called *uniform*. In the Main Theorem we will show that, for each uniform σ-ideal I of subsets of X, we have either $r(I) = 1$ or $r(I) = \omega_1$. It solves Mauldin's problem in negative.

Denote by $\mathcal{B}$ the family of all Borel subsets of X, and by Σ^0_α, Π^0_α (for $0 < \alpha < \omega_1$) - the subclasses of $\mathcal{B}$. A σ-ideal I is called Σ^0_2 *supported* if each $A \in I$ is contained in some $B \in I \cap \Sigma^0_2$. For a σ-ideal I, we define

$$I^* = \{A \subset X : (\exists B \in I \cap \Sigma^0_2)(A \subset B)\}.$$

Obviously, I^* is a Σ^0_2 supported σ-ideal and if I is a Σ^0_2 supported σ-ideal then $I = I^*$. Since the set of discontinuity points of an arbitrary function is of type F_σ, we have $C_I = C_{I^*}$ for each σ-ideal I, and thus the Baire order problem may be restricted to Σ^0_2 supported σ-ideals.

We will assume that I is a uniform σ-ideal of subsets of X. If $\mathcal{F}$ is a family of subsets of X then define $MGR(\mathcal{F})$ as the family of all subsets B of X such that for each $A \in \mathcal{F}$ the set $B \cap A$ is of the first category in A.

The following deep result plays a key role in the proof of our Main Theorem.

Proposition 1 [KS, thm 2] *Let I be a Σ^0_2 supported σ-ideal. Then precisely one of the following possibilities holds:*

(i) *$I = MGR(\mathcal{F})$ for a countable family $\mathcal{F}$ of closed subsets of X (moreover, it may be assumed that $\mathcal{F} = \{F_\gamma : \gamma < \alpha\}$ where $\alpha < \omega_1$ and $F_\gamma \subset F_\beta$ for $\beta < \gamma < \alpha$, and $F_{\gamma+1}$ is nowhere dense in F_γ for $\gamma < \alpha$);*

(ii) *there exists a homeomorphic embedding $\varphi : 2^\omega \times \omega^\omega \to X$ such that $\varphi[\{t\} \times \omega^\omega] \notin I$ for each $t \in 2^\omega$.* □

Proposition 2 *If a σ-ideal I satisfies condition (ii) of Proposition 1 then I has the following property:*

(M) *there exists a Borel function $f : X \to X$ such that $f^{-1}[\{x\}] \notin I$ for each $x \in X$.*

Define $R(I) = \min\{\alpha \leq \omega_1 : (\forall B \in \mathcal{B})(\exists A \in \Sigma^0_\alpha)(B \triangle A \in I)\}$ where $\Sigma^0_{\omega_1} = \mathcal{B}$ and $B \triangle A = (B \setminus A) \cup (A \setminus B)$. Observe that Σ^0_α can be replaced by Π^0_α in the above definition.

Proposition 3 [B3, Corollary 2.2] *If a σ-ideal I has the property (M) then $R(I) = \omega_1$.* □

Proposition 4 [M2, Thm 3] *For every σ-ideal I and each α, $0 < \alpha \leq \omega_1$, we have $f \in B_\alpha(I)$ if and only if there is $g \in B_\alpha$ such that $\{x \in X : f(x) \neq g(x)\} \in I^*$.* □

Proposition 5 [B2] *If I is a Σ^0_2 supported σ-ideal and $R(I) = \omega_1$ then $r(I) = \omega_1$.*

Using the above propositions we prove:

Theorem [BR] *If X is an uncountable Polish space and I is an uniform σ- ideal then either $r(I) = 1$ or $r(I) = \omega_1$.*

Remark. At this moment we do not know which values between 1 and ω_1 can be achieved by $r(I)$ when I is not uniform. Lately, some results in that direction have been obtained by I. Reclaw.

References

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