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THREE PROBLEMS IN EUCLIDEAN GEOMETRY

1 The Three Segment Problem

Let A, B and C be closed subsets of the open upper half-plane, $\mathbb{H}$ such that $A \cap B \cap C = \emptyset$. A point x on the boundary, $\partial\mathbb{H}$ of $\mathbb{H}$ is called an *ambiguous point* of the triple $(A < B < C)$ if there are arcs, α, β and γ in $\overline{\mathbb{H}}$ such that

- $\alpha \subset A,\ \beta \subset B,\ \gamma \subset C$
- $\alpha(1) = \beta(1) = \gamma(1) = x$
- $\alpha|_{[0,1)} \cup \beta|_{[0,1)} \cup \gamma|_{[0,1)} \subset \mathbb{H}$

If, in addition, the arcs α, β and γ are line segments (or rays) then x is called a *rectilinearly ambiguous point* of $(A < B < C)$. Questions concerning ambiguous points stem from the theory of cluster sets which was a hot topic in the 1950's and although a good deal is known about ambiguous point behavior, there is one tantalizing question which remains open.

Question 1 *Are there three closed subsets A, B and C of the open upper half-plane, $\mathbb{H}$ such that $A \cap B \cap C = \emptyset$ and every point of $\partial\mathbb{H}$ is a rectilinear ambiguous point relative to (A, B, C)?*

2 A Two Segment Problem

The second question is similar, but not entirely analogous. For this problem, $\mathbb{H}$ will denote the upper half of Euclidean three space and $\partial\mathbb{H}$ will denote its bounding plane. Let A and B be closed disjoint subsets of $\mathbb{H}$; a point $x \in \partial\mathbb{H}$ is an *ambiguous point* relative to (A, B) if there are arcs α and β such that

- $\alpha \subset A,\ \beta \subset B$

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- $\alpha(1) = \beta(1) = x$
- $\alpha|_{[0,1)} \cup \beta|_{[0,1)} \subset \mathbb{H}$

The question, then is this:

Question 2 *Are there disjoint closed subsets A and B of the open upper half-space, $\mathbb{H}$ such that every point of $\partial\mathbb{H}$ is a rectilinear ambiguous point relative to (A, B)?*

3 One Non–Segment Problem

In one small corner of Bob Vallin's survey [1] of results concerning shell porous sets there is an innocuous looking theorem, [1, Theorem 3.5] which caught my attention. (I've taken a bit of editorial privilege with Bob's material in what follows) A set $A \subset \mathbb{R}^2$ is a shell set if at each point of A there is a sequence of arbitrarily small annuli centered at the point which are contained in the complement of A. A σ–shell set is the countable union of shell sets.

Theorem 1 *If A is closed and σ-shell, then A is totally disconnected.*

I say innocuous because this theorem has a short proof and the result isn't in the main flow of ideas in Bob's paper. It is easy to see that

Theorem 2 *If A is closed and σ-shell, then the typical continuous function, $f : \mathbb{R} \to \mathbb{R}$ misses A.*

And it is also clear that Theorem 1 is a corollary to Theorem 2. But why *closed*? Is closed really necessary here? That is my third question.

Question 3 *If $A \subset \mathbb{R}^2$ is shell, then does the typical continuous function, $f : \mathbb{R} \to \mathbb{R}$ miss A?*

References

[1] B. Vallin, *Shell porous,porous, totally porous, hyperporous*, Real Anal. Exchange **17** No. 2 (1992–93), 294–320.