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SOME PROBLEMS CONCERNING ALMOST CONTINUOUS FUNCTIONS

A function $f\colon X \to Y$ is *almost continuous in the sense of Stallings* iff for every open set $U \subset X \times Y$ containing f, U contains a continuous function $g\colon X \to Y$ [12]. The notion of almost continuity was introduced in order to generalize the Brower fixed point theorem and has been studied later in many directions (see, e.g., [10] for the history of this notion).

In this note we shall consider only real functions defined on an interval, unless otherwise explicitly stated. The class of all almost continuous functions from X into Y will be denoted by $\mathcal{AC}(X,Y)$, or simply $\mathcal{AC}$ in case $X = Y = \mathbb{R}$. The class of all Darboux functions we will denote by $\mathcal{D}$. It is well-known that each almost continuous function defined on an interval is connected and therefore it possesses the Darboux property [12]. We want to present some problems concerning almost continuity.

1 Sums

For arbitrary families $\mathfrak{A}$ and $\mathfrak{B}$ of real functions let us define the following condition:

$U_a(\mathfrak{B};\mathfrak{A})$: there exists a $g\colon \mathbb{R} \to \mathbb{R}$ such that $f + g \in \mathfrak{A}$ whenever $f \in \mathfrak{B}$.

Let $a(\mathfrak{A})$ denote the least cardinal κ for which there exists a family $\mathfrak{B}$ of real functions such that $\mathrm{card}(\mathfrak{B}) = \kappa$ and $U_a(\mathfrak{B};\mathfrak{A})$ is false. We put $a(\mathfrak{A}) = 0$ if $U_a(\mathbb{R}^{\mathbb{R}};\mathfrak{A})$ holds.

It was remarked by Lindenbaum that $U_a(\{f\};\mathcal{D})$ holds for each real function f. This result was generalized by Fast in the following way: $U_a(\mathfrak{B};\mathcal{D})$ holds for each family $\mathfrak{B}$ of functions with $\mathrm{card}(\mathfrak{B}) \le 2^{\omega}$. In 1974, Kellum

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proved that Fast's theorem holds if we replace "$\mathcal{D}$" with "$\mathcal{AC}$" [5]. Hence $a(\mathcal{AC}) > 2^\omega$. On the other hand, it is easy to see that the condition $U_a(\mathbb{R}^\mathbb{R}; \mathcal{AC})$ is false. Indeed, for every function $g: \mathbb{R} \to \mathbb{R}$ there exists a function f such that $f + g$ does not have the Darboux property. Therefore $2^\omega < a(\mathcal{AC}) \leq 2^{2^\omega}$ and the assumption $(2^\omega)^+ = 2^{2^\omega}$ (in particular, it is a consequence of the Generalized Continuum Hypothesis) implies the equality $a(\mathcal{AC}) = 2^{2^\omega}$. We repeat the following question of [10].

Problem 1 *Can the equality $a(\mathcal{AC}) = 2^{2^\omega}$ be proved in ZFC*[1]

Recall that $U_a(\mathfrak{B}; \mathcal{AC})$ holds for some families $\mathfrak{B}$ of functions with cardinality 2^{2^ω}, e.g., if $\mathfrak{B}$ is either the family of all Lebesgue measurable functions or the family of all functions with the Baire property [10].

An analogous problem can be considered for the families of functions with a common bound. It is proved in [8] that the smallest cardinality of a family $\mathfrak{B}$ of real functions with a common bound for which there is no function $g: \mathbb{R} \to \mathbb{R}$ with the property that $f + g$ is a bounded Darboux function for all $f \in \mathfrak{B}$, is equal to the cofinality of the continuum. In particular, every bounded function $f: \mathbb{R} \to \mathbb{R}$ can be expressed as the sum of two bounded Darboux function. Thus we obtain the following question.

Problem 2 *Determine the smallest cardinality of a family $\mathfrak{B}$ of real functions with a common bound for which there is no $g: \mathbb{R} \to \mathbb{R}$ with the property that $f + g$ is bounded and almost continuous (or connected) for all $f \in \mathfrak{B}$. In particular, can we express each bounded function as the sum of two bounded almost continuous functions?*[2]

2 Additive almost continuous functions.

In [6], Kellum constructed an example of a discontinuous additive almost continuous function. Such functions have been also studied by Grande in [2].

Theorem 1 *Every additive function can be expressed as the sum of two additive almost continuous functions.*

Using standard methods this result can be generalized in the following way.

Theorem 2 *For each family $\mathfrak{B}$ of additive functions with* $\text{card}(\mathfrak{B}) \leq 2^\omega$ *there is an additive function $g: \mathbb{R} \to \mathbb{R}$ such that $f + g$ is almost continuous for all $f \in \mathfrak{B}$.*

[1] This question has been answered in the negative by K. Ciesielski and A. W. Miller, during the Joint US-Polish Workshop in Real Analysis, Łódź 1994 [1].

[2] P. Humke and U. Darji proved that each bounded function can be written as the sum of <u>three</u> bounded almost continuous functions, during the Joint US-Polish Workshop in Real Analysis, Łódź 1994 [3].

For a class $\mathfrak{A}$ of real functions let $\mathcal{M}_a(\mathfrak{A})$ denote the *maximal additive family* for $\mathfrak{A}$, i.e.,

$$\mathcal{M}_a(\mathfrak{A}) = \{g : f + g \in \mathfrak{A} \text{ for all } f \in \mathfrak{A}\}.$$

Recall that $\mathcal{M}_a(\mathcal{D})$ is the family of constant functions, and $\mathcal{M}_a(\mathcal{AC})$ consists of all continuous functions [4]. The problem to determine the maximal additive family for the class of all additive almost continuous functions is considered in [2]. It is proved there that the maximal additive family for the class of all additive almost continuous functions possessing the graph of the second category on the plane is equal to the family of all continuous additive functions, i.e., functions f of the form $f(x) = ax$ for some $a \in \mathbb{R}$. So we have the following two questions.

Problem 3 *Does there exist a discontinuous additive almost continuous (or connected) function whose graph is "small" in the sense of measure or category?*

(Recall that there are discontinuous additive Darboux functions possessing small graph both in the sense of measure and in the sense of category.)

Problem 4 *Characterize the maximal additive family for the class of all additive almost continuous functions.*

3 Compositions.

Obviously the class $\mathcal{D}$ is closed with respect to compositions. Thus the composition of almost continuous functions (from $\mathbb{R}$ into $\mathbb{R}$) has the Darboux property. On the other hand, there exists a function $f \in \mathcal{AC}(I, I)$ such that $f \circ f$ has no fixed point, whence it is not almost continuous [7]. The foregoing suggests the following problem (see [7], [9]).

Problem 5 *Is every Darboux function from $\mathbb{R}$ into $\mathbb{R}$ a composition of two (or more) almost continuous functions?*

The general answer to this question is not known to the authors. The following partial result is proved in [9].

Theorem 3 *Assume that the additivity of the ideal of the meager sets is equal to 2^ω. Then every Darboux function $f\colon \mathbb{R} \to \mathbb{R}$ with all level sets dense can be expressed as the composition of two almost continuous functions.*

We ask one more question.

Problem 6 *Can the above theorem be proved in ZFC?*

Finally let us recall that other problems concerning algebraic operations in the class of almost continuous functions have been presented in [10] and [11].

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