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KURZWEIL-HENSTOCK ABSOLUTE INTEGRABLE MEANS McSHANE INTEGRABLE

1 Introduction

The statement from the title is a well known theorem, proof of which often employs the concept of measurability, see e.g. [3]. Pfeffer [4] p. 45. asked for a proof which uses only the definitions of KH and McShane integrals (thereafter abbreviated to Mc-integrals). One has, of course, to take the word "only" with a grain of salt, how big this grain should be might be a matter of subjective judgement. In this paper we offer a proof which is reasonably simple, avoids any concept of measurability but needs a lot more than just those two definitions. Since the paper is methodical in character we state clearly all prerequisites from both KH and Mc integration theories which are needed. The use of the functions M_δ and m_δ below goes back to the original paper by Kurzweil [2] and our proof develops an idea of the proof of Theorem 5 in [1]. All functions appearing in this paper are real valued.

2 KH-Prerequisites

A partition of $[a, b]$ is a set of pairs $(x_i, [u_i, v_i])$ such that the intervals $[u_i, v_i]$ are non-overlapping, $x_i \in [u_i, v_i]$ and $\bigcup_1^n [u_i, v_i] = [a, b]$. If D is a partition of $[a, b]$,

$$D \equiv \{(x_i, [u_i, v_i]) : i = 1, 2 \ldots n\},$$

and $x_k \le c \le v_k$ then we denote

$$\sum_D f = \sum_{i=1}^{n} f(x_i)(v_i - u_i)$$

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$$\text{and} \quad \sum_D {}_a^c f = \sum_{i=1}^{k-1} f(x_i)(v_i - u_i) + f(x_k)(c - u_k).$$

If D is a δ-fine partition we write $D \ll \delta$. If f is integrable on a compact interval $[a, b]$, $a < x \leq b$, then for every $\varepsilon > 0$ there is a $\delta : [a,b] \to (0,\infty)$ such that both functions

$$M_\delta(x) = \sup\{\sum_D {}_a^x f : D \ll \delta\}, \quad M_\delta(a) = 0 \tag{1}$$

$$m_\delta(x) = \inf\{\sum_D {}_a^x f : D \ll \delta\}, \quad m_\delta(a) = 0 \tag{2}$$

are well defined on $[a, b]$ and they are major and minor functions in the sense that

$$f(x)(v-u) \leq M_\delta(v) - M_\delta(u) \quad \text{and} \quad f(x)(v-u) \geq m_\delta(v) - m_\delta(u) \tag{3}$$

whenever $a \leq x - \delta(x) < u \leq x \leq v < x + \delta(x) \leq b$. Moreover $M_\delta - m_\delta$ is increasing and

$$M_\delta(b) - m_\delta(b) < \varepsilon. \tag{4}$$

If f is non-negative then both M_δ and m_δ are increasing.

3 Mc Prerequisites

Mc-integrable functions form a vector space over the reals and also a vector lattice. Monotonic functions are Mc-integrable over compact intervals. If for every positive ε there are Mc-integrable functions H and h such that

$$\begin{aligned} h(x) \leq f(x) \leq H(x) \qquad & \text{for} \quad a \leq x \leq b, \\ \mathrm{Mc}\int_a^b H - \mathrm{Mc}\int_a^b h \; < \; & \varepsilon \end{aligned}$$

then f is Mc-integrable (and $\mathrm{Mc}\int_a^b f = \inf_H \mathrm{Mc}\int_a^b H = \sup_h \mathrm{Mc}\int_a^b h$). Finally Fatou's[1] lemma is needed: If F_n are non-negative, Mc-integrable and $\{\mathrm{Mc}\int_a^b F_n\}$ bounded then $\liminf F_n$ is Mc-integrable and

$$\mathrm{Mc}\int_a^b \liminf_{n\to\infty} F_n \leq \liminf_{n\to\infty} \mathrm{Mc}\int_a^b F_n.$$

[1] Fatou's lemma is a consequence of the monotone convergence theorem which is valid for the Mc-integral.

4 The Proof

It is sufficient to consider a non-negative f. For convenience sake we set $f(x) = 0$ for $x \geq b$ and extend m_δ and M_δ naturally, i.e. $m_\delta(x) = m_\delta(b)$ and $M_\delta(x) = M_\delta(b)$ for $x \geq b$. Since f is KH-integrable we obtain for every positive ε functions m_δ and M_δ as in (2) and (1). Both of these functions are Mc-integrable because they are increasing. Let

$$\begin{aligned} H_n(x) &= n(M_\delta(x+\frac{1}{n}) - M_\delta(x)), \\ h_n(x) &= n(m_\delta(x+\frac{1}{n}) - m_\delta(x)), \\ H_\infty &= \liminf_{n\to\infty} H_n, \\ h^\infty &= \limsup_{n\to\infty} h_n. \end{aligned}$$

The functions H_n and h_n are Mc-integrable, non-negative and their integrals are uniformly bounded, we have e.g.[2]

$$0 \leq \mathrm{Mc}\int_a^b H_n \leq M_\delta(b). \tag{5}$$

By Fatou's lemma H_∞ is Mc-integrable. To obtain integrability of h^∞ we apply Fatou's lemma to the sequence $\{(H_\infty - h_n)^+\}$. For $n > 1/\delta(x)$ we have $h_n(x) \leq f(x) \leq H_\infty(x)$ and consequently

$$\liminf_{n\to\infty} (H_\infty - h_n)^+ = H_\infty - h^\infty.$$

Fatou's lemma now assures the integrability of $H_\infty - h^\infty$. By (3) the function f is trapped between Mc-integrable H_∞ and h^∞. It remains to show that $\mathrm{Mc}\int_a^b(H_\infty - h^\infty) < \varepsilon$. To do this we repeatedly use Fatou's lemma.

First we have

$$\begin{aligned} H_\infty - h^\infty &\leq \liminf_{n\to\infty} (H_n - h_n), \\ \mathrm{Mc}\int_a^b (H_\infty - h^\infty) &\leq \mathrm{Mc}\int_a^b \liminf_{n\to\infty} (H_n - h_n), \end{aligned}$$

and then

$$\mathrm{Mc}\int_a^b \liminf_{n\to\infty} (H_n - h_n) \leq \liminf_{n\to\infty} \mathrm{Mc}\int_a^b (H_n - h_n).$$

[2]No substitution theorem is needed, $\mathrm{Mc}\int_a^b M_\delta(x+1/n)dx = \mathrm{Mc}\int_{a+1/n}^{b+1/n} M_\delta(x)dx$ follows easily from the definition of the Mc-integral.

By analog of (5) and with the use of (4)

$$\liminf_{n\to\infty} \mathrm{Mc}\int_a^b (H_n - h_n) \leq \; M_\delta(b) - m_\delta(b) < \varepsilon.$$

Consequently

$$\mathrm{Mc}\int_a^b (H_\infty - h^\infty) < \varepsilon.$$

The proof is complete.

References

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