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INEQUALITIES OF MINKOWSKI'S TYPE

Abstract

Let $f : [a,b] \to \mathbb{R}$ be a nonnegative and nondecreasing function and let $x_i : [a,b] \to \mathbb{R}$ $(i = 1, \dots, n)$ be nonnegative and nondecreasing functions with continuous first derivative. If $p > 1$, then

$$\left(\int_b^a \left((\sum_{i=1}^n x_i(t))^p\right)' f(t)dt\right)^{1/p} \geq \sum_{i=1}^n \left(\int_a^b (x_i^p(t))' f(t)dt\right)^{1/p}.$$

If f is a nonincreasing function and $x_i(a) = 0$ for all $i = 1, \dots, n$, then the reverse inequality is valid.

1 Introduction

In [1] H. Alzer gave the following theorem:

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be a nonnegative and increasing function and let $g : [a,b] \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ be nonnegative and increasing functions with continuous first derivatives. If $g(a) = h(a)$ and $g(b) = h(b)$, then*

$$(1) \quad \left(\int_a^b \left(\sqrt{g(x)h(x)}\right)' f(x)\,dx\right)^2 \geq \int_a^b g'(x)f(x)dx \int_a^b h'(x)f(x)\,dx.$$

He showed that the previous result is an extension of the inequality which can be found in [3, Vol. I page 83]. In fact, if we substitute in (1): $a = 0$, $b = 1$, $g(x) = x^{2u+1}$, $h(x) = x^{2v+1}$ where u, $v > 0$, then we have the theorem:

Theorem 2 *Let $f : [0,1] \to \mathbb{R}$ be a nonnegative and increasing function. If u and v are nonnegative real numbers, then*

(2)
$$\left(\int_0^1 x^{u+v} f(x)\,dx\right)^2 \geq \left(1 - \left(\frac{u-v}{u+v+1}\right)^2\right) \int_0^1 x^{2u} f(x)\,dx \int_0^1 x^{2v} f(x)\,dx.$$

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Also, in the above-mentioned famous book [3, Vol. II] one can find the reverse inequality of (2):

Theorem 3 *Let $f : [0, \infty) \to \mathbb{R}$ be a nonnegative and decreasing function. If u and v are nonnegative real numbers, then*

(3)
$$\left(\int_0^\infty x^{u+v} f(x)\,dx\right)^2 \le \left(1 - \left(\frac{u-v}{u+v+1}\right)^2\right) \int_0^\infty x^{2u} f(x)\,dx \int_0^\infty x^{2v} f(x)\,dx.$$

(If the above-mentioned improper integrals exist.)

In [1] the well-known inequality between arithmetic and geometric means is used for proving inequality (1), but that method can not be used for proving the reverse of (1).

In this paper, we will give a generalization of (1), (2) and (3), and obtain the similar treatment for the inequality (1) and its reverse.

2 Main Results

Theorem 4 *Let $f : [a, b] \to \mathbb{R}$ be a nonnegative and nondecreasing function and let $x_i : [a, b] \to \mathbb{R}$ $(i = 1, \ldots, n)$ be nonnegative and nondecreasing functions with continuous first derivative. If $p > 1$, then*

$$(4) \qquad \left(\int_b^a \left((\sum_{i=1}^n x_i(t))^p\right)' f(t)\,dt\right)^{1/p} \ge \sum_{i=1}^n \left(\int_a^b (x_i^p(t))' f(t)\,dt\right)^{1/p}.$$

If f is a nonincreasing function and $x_i(a) = 0$ for all $i = 1, \ldots, n$, then the reverse inequality of (4) is valid.

PROOF. Suppose that f is a nondecreasing function. Using integration by parts and Minkowski's inequality for integrals we obtain

$$(5) \qquad \left(\int_b^a \left((\sum_{i=1}^n x_i(t))^p\right)' f(t)\,dt\right)^{1/p} =$$

$$= \left(f(b)(\sum_{i=1}^n x_i(b))^p - f(a)(\sum_{i=1}^n x_i(a))^p - \int_b^a (\sum_{i=1}^n x_i(t))^p df(t)\right)^{1/p} \ge$$

$$\ge \left(f(b)(\sum_{i=1}^n x_i(b))^p - f(a)(\sum_{i=1}^n x_i(a))^p - \left(\sum_{i=1}^n \left(\int_a^b x_i^p(t)\,df(t)\right)^{1/p}\right)^p\right)^{1/p}.$$

Recall Bellman's inequality [2, page 118]:

If $a_{ij} \geq 0, w_j \geq 0$, $i = 1, \ldots, n, j = 1, \ldots, m$ and if $p > 1$ satisfies

$$w_1 a_{i1}^p - w_2 a_{i2}^p - \ldots - w_m a_{im}^p \geq 0 \text{ for all } i = 1, \ldots, n,$$

then

$$\left(\sum_{i=1}^{n} (w_1 a_{i1}^p - w_2 a_{i2}^p - \ldots - w_m a_{im}^p)^{1/p} \right)^p \leq \tag{6}$$
$$\leq w_1 (\sum_{i=1}^{n} a_{i1})^p - w_2 (\sum_{i=1}^{n} a_{i2})^p - \ldots - w_m (\sum_{i=1}^{n} a_{im})^p.$$

Setting in (2) :

$$m = 3, \ \ w_1 = f(b), \ \ w_2 = f(a), \ \ w_3 = 1,$$

$$a_{i1} = x_i(b), \ a_{i2} = x_i(a), \ a_{i3} = \left(\int_a^b x_i(t)^p df(t) \right)^{1/p} \quad \text{for } i = 1, \ldots, n,$$

we get that (5) is greater than or equal to

$$\sum_{i=1}^{n} \left(f(b) x_i^p(b) - f(a) x_i^p(a) - \int_a^b x_i^p(t)\, df(t) \right)^{1/p} =$$
$$= \sum_{i=1}^{n} \left(\int_a^b (x_i(t)^p)' f(t)\, dt \right)^{1/p}.$$

If f is a nonincreasing function, then the proof is similar to the previous one with the only difference that instead of Bellman's inequality, Minkowski's inequality for the discrete case is used.□

Corollary 1 *If f, g, h are nondecreasing and nonnegative functions and if g and h have continuous first derivatives, then*

$$\left(\int_a^b \left(\sqrt{g(x)h(x)} \right)' f(x)\, dx \right)^2 \geq \int_a^b g'(x) f(x)\, dx \int_a^b h'(x) f(x)\, dx. \tag{7}$$

If $g(a) = h(a) = 0$ and if f is a nonincreasing function, then the reverse inequality holds.

PROOF. It is a simple consequence of Theorem 4 in the case when $n = 2$, $p = 2$ and $x_1(t) = \sqrt{g(t)}$, $x_2(t) = \sqrt{h(t)}$.□

Remark 1 If f is nondecreasing, the statement in Corollary 1 is similar to the statement of Theorem 1, but $g(a) = h(a)$ and $g(b) = h(b)$ are not required. Also, in the corollary we give the reverse inequality.

Remark 2 Setting in (7) : $a = 0,\ b = 1,\ g(t) = t^{2u+1},\ h(t) = t^{2v+1}$ for $a, b > -\frac{1}{2}$ and if f is nondecreasing we obtain inequality (2). And taking in the reverse of (7) : $a = 0,\ b = B,\ g(t) = t^{2u+1},\ h(t) = t^{2v+1}$ we have

(8)
$$\left(\int_0^B x^{u+v} f(x)\,dx\right)^2 \le \left(1 - \left(\frac{u-v}{u+v+1}\right)^2\right) \int_0^B x^{2u} f(x)\,dx \int_0^B x^{2v} f(x)\,dx,$$

where f is a nonincreasing function. If the improper integrals $\int_0^\infty x^{u+v} f(x)\,dx$, $\int_0^\infty x^{2u} f(x)\,dx$, $\int_0^\infty x^{2v} f(x)\,dx$ exist, then from (8) we obtain (3).

The following theorem deals with derivatives of higher order.

Theorem 5 *Let $p > 1$ and let $f,\ x_i : [a,b] \to \mathbb{R}$ $(i = 1, \ldots, m)$ be nonnegative functions with continuous derivative of the n-th order which satisfy the conditions*

1^o $(-1)^k f^{(k)}(b) \ge 0$ *for* $k = 1, \ldots, n-1$,

2^o $x_i^{(k)}(a) = 0$ *for* $k = 0, \ldots, n-1,\ i = 1, \ldots, m$ *and* $(\sum_{i=1}^m x_i^p(b))^{(k)} \ge 0$ *for* $k = 1, \ldots, n-1$,

3^o $(x_i^p(t))^{(n)} \ge 0$ *for* $t \in [a,b]$ *and* $i = 1, \ldots, m$,

4^o $(-1)^n f^{(n)}(t) \ge 0$ *for* $t \in [a,b]$.

If

$$(9)\quad x_i^{(k)}(b) \quad = \quad x_j^{(k)}(b) \text{ for all } i, j \in \{1, 2, \ldots, m\} \text{ and } k = 0, 1, \ldots, n-1,$$

then

$$(10)\quad \int_a^b \left((\sum_{i=1}^m x_i(t))^p\right)^{(n)} f(t)\,dt \ \le\ \left(\sum_{i=1}^m \left(\int_a^b (x_i(t)^p)^{(n)} f(t)\,dt\right)^{1/p}\right)^p .$$

If the conditions in (9) are not satisfied, then

$$(11)\quad \int_a^b \left((\sum_{i=1}^m x_i(t))^p\right)^{(n)} f(t)\,dt \le \Delta + \left(\sum_{i=1}^m \left(\int_a^b (x_i^p(t))^{(n)} f(t)\,dt\right)^{1/p}\right)^p$$

where

$$\Delta = \sum_{k=0}^{n-1}(-1)^k f^{(k)}(t)\left(\left(\left(\sum_{i=1}^m x_i(t)\right)^p\right)^{(n-k-1)}\right.$$
$$\left. - \left(\sum_{i=1}^m \left((x_i^p(t))^{(n-k-1)}\right)^{1/p}\right)^p\right)\Big|_{t=b}.$$

If the inequalities in conditions 1^o *and* 4^o *are reversed, then reverse inequalities of (11) and (10) hold.*

The inequality (11) may be obtained via Minkowski's inequality as in the second proof of Theorem 4, after integrating by parts n times. The details are left to the reader. To check the validity of inequality (10) we need the following lemma.

Lemma 1 *If* x_i $(i = 1, \dots, m)$ *satisfy the assumptions of Theorem 5 and if* $x_i^{(k)}(b) = x_j^{(k)}(b) = B_k$ *for* $i, j \in \{1, \dots, m\}$ *and* $k = 0, \dots, n-1$, *then*

$$\left(\left(\left(\sum_{i=1}^m x_i(t)\right)^p\right)^{(k)}\right)^{1/p}\Bigg|_{t=b} = \sum_{i=1}^m \left((x_i^p(t))^{(k)}\right)^{1/p}\Bigg|_{t=b}.$$

PROOF. Let us first consider the k-th derivative of the function y^p where y is an arbitrary function having a k-th derivative. We will prove by induction that there exist functions $\phi_k^{[p]}$ such that

$$(y^p)^{(k)} = \phi_k^{[p]}(y, y', \dots, y^{(k)})$$

and $\phi_k^{[p]}$ is homogeneous of the order p. For $k = 1$ we have

$$(y^p)' = py^{p-1}y' = \phi_1^{[p]}(y, y')$$

and

$$\phi_1^{[p]}(ny, ny') = p(ny)^{p-1}ny' = n^p \cdot \phi_1^{[p]}(y, y') .$$

Suppose that the statement is valid for any $j < k+1$. Then using Leibniz's rule we get

$$(y^p)^{(k+1)} = (py^{p-1} \cdot y')^{(k)} = p\sum_{i=0}^m \binom{k}{j}(y^{p-1})^{(j)}(y')^{(k-j)} =$$

$$= p\sum_{j=0}^{k}\binom{k}{j}\phi_j^{[p-1]}(y, y', \ldots, y^{(j)})y^{(k-j+1)} = \phi_{k+1}^{[p]}(y, y', \ldots, y^{(k+1)})$$

and

$$\phi_{k+1}^{[p]}(ny, ny', \ldots, ny^{(k+1)}) = p\sum_{j=0}^{k}\binom{k}{j}\phi_j^{[p-1]}(ny, ny', \ldots, ny^{(j)})(ny)^{(k-j+1)} =$$

$$= p\sum_{j=0}^{k}\binom{k}{j}n^{p-1}\phi_j^{[p-1]}(y, y', \ldots, y^{(j)})ny^{(k-j+1)} =$$

$$= n^p\phi_{k+1}^{[p]}(y, y', \ldots, y^{(k+1)}).$$

Now if $y = x_i$, then

$$(x_i^p(t))^{(k)}\Big|_{t=b} = \phi_k^{[p]}(x_i, x_i', \ldots, x_i^{(k)})\Big|_{t=b} = \phi_k^{[p]}(B_0, B_1, \ldots, B_k)$$

and for $y = \sum_{i=1}^m x_i$ we get

$$\left(\left(\sum_{i=1}^m x_i(t)\right)^p\right)^{(k)}\Bigg|_{t=b} = \phi_k^{[p]}\left(\sum_{i=1}^m x_i, \sum_{i=1}^m x_i', \ldots, \sum_{i=1}^m x_i^{(k)}\right)\Bigg|_{t=b} =$$

$$= \phi_k^{[p]}(mB_0, mB_1, \ldots, mB_k) = m^p\phi_k^{[p]}(B_0, B_1, \ldots, B_k).$$

So,

$$\left(\left((\sum_{i=1}^m x_i(t))^p\right)^{(k)}\right)^{1/p}\Big|_{t=b} = \left(m^p\phi_k^{[p]}(B_0, \ldots, B_k)\right)^{1/p} =$$

$$= m\left(\phi_k^{[p]}(B_0, \ldots, B_k)\right)^{1/p} = \sum_{i=1}^m\left(\phi_k^{[p]}(B_0, \ldots, B_k)\right)^{1/p} = \sum_{i=1}^m\left((x_i^p(b))^{(k)}\right)^{1/p}$$

and the proof has been established.□

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