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DENSITY TOPOLOGIES FOR PRODUCTS OF σ-IDEALS

Abstract

According to Wilczyński's scheme, we consider density points for product σ-ideals $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$. We get the properties analogous to Lebesgue density theorem and create the respective topologies.

1 Introduction

In [W] (see also [PWW]) Wilczyński introduced an abstract notion of a density point. It was associated with a given σ-ideal in a σ-algebra of sets in $\mathbb{R}$, invariant with respect to linear transformations. If the σ-ideal consists of null sets in the σ-algebra of Lebesgue measurable sets, this concept yields the usual density points. It is important that Wilczyński's idea works for category and then the analogue of the density topology, called the $\mathcal{I}$-density topology, can be defined. The theory of $\mathcal{I}$-density in the category case was developed in many papers (see [W], [CLO]). Similar considerations were carried out if $\mathcal{I}$ is the σ-ideal of plane meager sets (see e.g. [W], [CW]). In this article, we give new applications of Wilczyński's definition by the use of products of σ-ideals.

Let us recall some basic notation. Assume that $\mathcal{I} \neq \{\emptyset\}$ is a σ-ideal of subsets of $\mathbb{R}$ and let $\mathcal{S}$ be a σ-algebra containing $\mathcal{I}$. Suppose that $\mathcal{I}$ and $\mathcal{S}$ are invariant with respect to linear transformations.

For $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}^2$ and $s, t \in \mathbb{R}$, we denote

$$pr_1(s,t) = s, \qquad pr_2(s,t) = t,$$

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$$A \pm s = \{x \pm s : x \in A\}, \qquad s \cdot A = \{sx : x \in A\},$$
$$B \pm \langle s,t\rangle = \{\langle x \pm s,\ y \pm t\rangle : \langle x,y\rangle \in B\},$$
$$\langle s,t\rangle \cdot B = \{\langle sx, ty\rangle : \langle x,y\rangle \in B\}.$$

Definition 1.1 (See [W], [CLO, p.17].) *We say that:*

(a) 0 is an $\mathcal{I}$-density point of $A \in \mathcal{S}$ if, for each increasing sequence $\{n_i\}$ of positive integers, there is a subsequence $\{n_{i_k}\}$ such that

$$\limsup_{k\to\infty}((-1,1)\backslash n_{i_k}A) \in \mathcal{I};$$

(b) $x_0 \in \mathbb{R}$ is an $\mathcal{I}$-density point of $A \in \mathcal{S}$ if 0 is an $\mathcal{I}$-density point of $A - x_0$;

(c) x_0 is an $\mathcal{I}$-dispersion point of $A \in \mathcal{S}$ if it is an $\mathcal{I}$-density point of $\mathbb{R}\backslash A$.

Observe that if $\mathbb{R} \notin \mathcal{I}$ (which will always be assumed) and x_0 is an $\mathcal{I}$-density point of $A \in \mathcal{S}$, then x_0 is an accumulation point of A (note that from $\mathbb{R} \notin \mathcal{I}$ and the invariance of $\mathcal{I}$ under all translations it follows that $U \notin \mathcal{I}$ for any open interval U).

Analogous definitions can be formulated if $\mathcal{I}$ is a σ-ideal of sets in $\mathbb{R}^2$, $\mathcal{S}$ is the corresponding σ-algebra and both the families $\mathcal{I}$ and $\mathcal{S}$ are invariant with respect to all translations in $\mathbb{R}^2$ and the mappings of the form $\langle x,y\rangle \mapsto \langle ax, ay\rangle$ where $a \neq 0$. Then $\langle 0,0\rangle$ is called an *$\mathcal{I}$-density point* of $A \in \mathcal{S}$ if

$$\forall_{\{n_i\}}\exists_{\{n_{i_k}\}} \limsup_{k\to\infty}((-1,1)^2\backslash\langle n_{i_k}, n_{i_k}\rangle A) \in \mathcal{I}.$$

If a pair $\langle \mathcal{S},\mathcal{I}\rangle$ is given (for sets in $X = \mathbb{R}$ or $X = \mathbb{R}^2$) and $A \in \mathcal{S}$, we denote by $\varphi_{\mathcal{I}}(A)$ the set of all $\mathcal{I}$-density points of A. When $A, B \in \mathcal{S}$ and the symmetric difference $A \triangle B$ is in $\mathcal{I}$, we write $A \sim B$. The operator $\varphi_{\mathcal{I}}$ (further denoted by φ) fulfils (for any $A, B \in \mathcal{S}$) the following conditions:

(1) if $A \sim B$, then $\varphi(A) = \varphi(B)$,

(2) $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$,

(3) $\varphi(\emptyset) = \emptyset$, $\varphi(X) = X$.

The proof is the same as that given in [CLO, Lemma 2.3.1]. Any operator satisfying the above conditions and, additionally,

(4) $\varphi(A) \sim A$

(for each $A \in \mathcal{S}$) is called an *operator of lower density*.

Let us assume that (4) holds, and that each $E \subseteq X$ has its $\mathcal{S}$-measurable cover (i.e., a set $A \in \mathcal{S}$ satisfying $E \subseteq A$ and $\mathcal{P}(A \setminus E) \cap \mathcal{S} \subseteq \mathcal{I}$). Then one can repeat the classical proof showing that the family

$$\tau_{\mathcal{I}} = \{A \in \mathcal{S} : A \subseteq \varphi(A)\}$$

forms a topology (see [LMZ, Prop.6.37] and also [O, Th.22.4]); $\tau_{\mathcal{I}}$ is called the the $\mathcal{I}$-density topology. For $\mathcal{I}$ = the Lebesgue null sets we get the usual density topology and for $\mathcal{I}$ = the meager sets – its category analogue proposed by Wilczyński.

Notice that condition (4) fails to hold for some couples $\langle \mathcal{S}, \mathcal{I} \rangle$. Indeed, the following example was brought to us by K. Ciesielski. Consider $\langle \mathcal{P}(\mathbb{R}), \mathcal{I} \rangle$ where $\mathcal{I}$ is either the ideal of meager sets or the ideal of null sets. Then, for a Hamel base being a Bernstein set, we easily obtain $\varphi(\mathbb{R} \setminus A) = \mathbb{R}$, so (4) is false. Some other examples can be found in [BHWW].

The existense of $\mathcal{S}$-measurable covers is ensured (cf. e.g. [F1, Lemma 1H(b)]) by the so-called *countable chain condition* (in short, ccc) which means that each disjoint subfamily of $\mathcal{S} \setminus \mathcal{I}$ is countable. We shall give new examples in which (4) holds and ccc is valid simultaneously.

The following lemma states that, when (1), (2), (3) hold, condition (4) is automatically true if one assumes its weakened version. (It will be used in the proofs of Propositions 3.3 and 4.3.)

Lemma 1.2 *Assume that (1), (2), (3) hold for any $A, B \in \mathcal{S}$ and let $\mathcal{F} \subset \mathcal{S}$ be a family such that*

$$\forall_{A \in \mathcal{S}} \exists_{B \in \mathcal{F}} \;\; A \sim B,$$
$$\forall_{B \in \mathcal{F}} \;\; B \setminus \varphi(B) \in \mathcal{I}.$$

Then (4) holds for each $A \in \mathcal{S}$.

PROOF. Let $A \in \mathcal{S}$. From the assumptions we easily derive that $A \setminus \varphi(A) \in \mathcal{I}$. Choose $B \in \mathcal{F}$ such that $X \setminus A \sim B$. Thus $A \cap B \in \mathcal{I}$. Hence

$$\varphi(A) \cap \varphi(B) = \varphi(A \cap B) = \varphi(\emptyset) = \emptyset.$$

Consequently, $\varphi(A) \subseteq X \setminus \varphi(B)$ and thus we get

$$\varphi(A) \setminus A \sim \varphi(A) \cap B \subseteq B \setminus \varphi(B) \in \mathcal{I}. \quad \square$$

Let $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R}^2)$ denote the families of Borel sets in $\mathbb{R}$ and $\mathbb{R}^2$, respectively. The ideals of Lebesgue null sets and of meager sets in $\mathbb{R}$ will be written as $\mathbb{L}$ and $\mathbb{K}$, respectively. If $\mathcal{I}$ is a fixed σ-ideal of sets in $\mathbb{R}$ (or in $\mathbb{R}^2$), we denote by $\mathcal{S}(\mathcal{I})$ the σ-algebra generated by $\mathcal{I} \cup \mathcal{B}(\mathbb{R})$ (or $\mathcal{I} \cup \mathcal{B}(\mathbb{R}^2)$).

For $E \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}$, we denote

$$E_x = \{y \in \mathbb{R} : \langle x, y \rangle \in E\}.$$

Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals of subsets of $\mathbb{R}$. We define

$$(\nabla) \qquad \mathcal{I} \times \mathcal{J} = \{E \subseteq \mathbb{R}^2 : \exists_{B \in \mathcal{B}(\mathbb{R}^2)} (E \subseteq B \& \{x \in \mathbb{R} : B_x \notin \mathcal{J}\} \in \mathcal{I})\}.$$

Then $\mathcal{I} \times \mathcal{J}$ forms a σ-ideal of plane sets, called the product of $\mathcal{I}$ and $\mathcal{J}$. Observe that if $\mathcal{I} = \mathcal{J} = \mathbb{L}$ or $\mathcal{I} = \mathcal{J} = \mathbb{K}$, then $\mathcal{I} \times \mathcal{J}$ is (by the Fubini theorem and by the Kuratowski-Ulam theorem) equal to the σ-ideal of plane Lebesgue null sets or of plane meager sets, respectively. The mixed products $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ form new interesting σ-ideals (cf. [CP], [G], [M]). In the paper we investigate $\mathcal{I} \times \mathcal{J}$-density points, in particular, $\mathbb{L} \times \mathbb{K}$- and $\mathbb{K} \times \mathbb{L}$-density points.

Let us explain why we do not define $\mathcal{I} \times \mathcal{J}$ simply as

$$(\nabla_*) \qquad \{E \subseteq \mathbb{R}^2 : \{x \in \mathbb{R} : E_x \notin \mathcal{J}\} \in \mathcal{I}\}.$$

Namely, we want to associate the product of σ-ideals with Borel sets to avoid the existence of pathological sets in $\mathcal{I} \times \mathcal{J}$ which cannot be covered by Borel ones from $\mathcal{I} \times \mathcal{J}$ (cf. [M, Th.1.3]). It is the same reason why one should assume the measurability of a plane set to convert the Fubini theorem. (See [O, Th.14.3].) Note that, in connections of $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ with forcing [CP] and Boolean algebras [G], the version engaging Borel sets is natural. Observe that, assuming definition (∇) we have $E \in \mathcal{S}(\mathcal{I} \times \mathcal{J})$ iff $\{x \in \mathbb{R} : E_x \notin \mathcal{J}\} \in \mathcal{I}$, so, for sets from $\mathcal{S}(\mathcal{I} \times \mathcal{J})$, conditions (∇) and (∇_*) mean the same. Another reason that we use (∇) is the referring to the literature where that version is supposed (Lemmas 3.1 and 4.1 below).

2 General Case

Assume that $\mathcal{I}$ and $\mathcal{J}$ are σ-ideals of subsets of $\mathbb{R}$, invariant with respect to linear transformations. Consider the σ-algebras $\mathcal{S}(\mathcal{I})$ and $\mathcal{S}(\mathcal{J})$ and associate the σ-algebra $\mathcal{S}(\mathcal{I} \times \mathcal{J})$ with the product $\mathcal{I} \times \mathcal{J}$. Observe that, for any $E \in \mathcal{S}(\mathcal{I} \times \mathcal{J})$, we have $\{x \in \mathbb{R} : E_x \notin \mathcal{S}(\mathcal{J})\} \in \mathcal{I}$. This follows from the definition of $\mathcal{I} \times \mathcal{J}$ and from the fact that $B_x \in \mathcal{B}(\mathbb{R})$ for any $B \in \mathcal{B}(\mathbb{R}^2)$ and $x \in \mathbb{R}$.

Lemma 2.1 *Let $E \subseteq \mathbb{R}^2$ and $s, t, x \in \mathbb{R}$. Then*

(a) $(E + \langle s,t\rangle)_x = E_{x-s} + t$,

(b) $(\langle s,t\rangle \cdot E)_x = t(E_{x/s})$ *for* $s \neq 0$,

(c) if $E \in \mathcal{I} \times \mathcal{J}$, *then* $E + \langle s,t\rangle \in \mathcal{I} \times \mathcal{J}$,

(d) if $E \in \mathcal{I} \times \mathcal{J}$, *then* $\langle s,t\rangle \cdot E \in \mathcal{I} \times \mathcal{J}$.

PROOF. Statements (a) and (b) can be checked directly. Conditions (c), (d) follow from (a), (b), respectively, by the use of the invariance of $\mathcal{I}$ and $\mathcal{J}$ with respect to linear transformations. □

Observe that, for $E \in \mathcal{S}(\mathcal{I} \times \mathcal{J})$, we have $E \in \mathcal{I} \times \mathcal{J}$ iff $\{x \in \mathbb{R} : E_x \notin \mathcal{J}\} \in \mathcal{I}$. According to Lemma 2.1 and the remarks at the end of Section 1, the general definition (given in Section 1) applied to the case $\mathcal{I} \times \mathcal{J}$ reads as follows

Definition 2.2 *The point $\langle 0,0\rangle$ is an $\mathcal{I} \times \mathcal{J}$-density point of $E \in \mathcal{S}(\mathcal{I} \times \mathcal{J})$ iff*

$$\forall_{\{n_i\}} \exists_{\{n_{i_k}\}} \exists_{A \in \mathcal{I}} \forall_{x \in (-1,1)\setminus A} \quad \limsup_{k\to\infty}((-1,1)\setminus n_{i_k}(E_{x/n_{i_k}})) \in \mathcal{J}.$$

In the standard way (cf. Definition 1.1(b)) we extend this definition to the case when $\langle x_0, y_0\rangle$ is taken instead of $\langle 0,0\rangle$. (Then $(E_{(x/n_{i_k})+x_0} - y_0)$ appears, by Lemma 2.1 (a).)

Proposition 2.3 *Assume that $\langle 0,0\rangle$ is an $\mathcal{I} \times \mathcal{J}$-density point of a set $E \in \mathcal{S}(\mathcal{I} \times \mathcal{J})$. Then 0 is an $\mathcal{I}$-density point of each set $D \in \mathcal{S}(\mathcal{I})$ containing $pr_1 E$ and 0 is a $\mathcal{J}$-density point of each set $H \in \mathcal{S}(\mathcal{J})$ containing $pr_2 E$.*

PROOF. To show the first assertion suppose to the contrary that 0 is not an $\mathcal{I}$-density point of D. Thus there is a sequence $\{n_i\}$ such that, for each subsequence $\{n_{i_k}\}$, we have

$$B = \limsup_{k\to\infty}((-1,1)\setminus n_{i_k} D) \notin \mathcal{I}.$$

Since $\langle 0,0\rangle$ is an $\mathcal{I} \times \mathcal{J}$-density point of E, we can choose a subsequence $\{n_{i_k}\}$ and a set $A \in \mathcal{I}$ such that, for each $x \in (-1,1)\setminus A$, we have

$$(\triangle) \qquad \limsup_{k\to\infty}((-1,1)\setminus n_{i_k} E_{x/n_{i_k}})) \in \mathcal{J}.$$

Fix a point $x \in B\setminus A$. Thus there is a subsequence $\{n_{i_{k_j}}\}$ of $\{n_{i_k}\}$ such that $\frac{x}{n_{i_{k_j}}} \notin D$ for every j. Since $D \supseteq pr_1 E$, it follows that $E_{x/n_{i_{k_j}}} = \emptyset$ for every j and, consequently,

$$\limsup_{k\to\infty}((-1,1)\setminus n_{i_k} E_{x/n_{i_k}}) = (-1,1)$$

which contradicts $(\triangle)$.

To get the second assertion, consider any sequence $\{n_i\}$. By the assumption, we can pick a subsequence $\{n_{i_k}\}$ and a set $A \in \mathcal{I}$ such that, for each $x \in (-1,1)\backslash A$, condition $(\triangle)$ holds. Since $E_{x/n_{i_k}} \subseteq H$, we have

$$\limsup_{k\to\infty}((-1,1)\backslash n_{i_k}H) \subseteq \limsup_{k\to\infty}((-1,1)\backslash n_{i_k}E_{x/n_{i_k}}) \in \mathcal{J}. \quad \square$$

Proposition 2.4 *If 0 is an $\mathcal{I}$-density point of $A \in \mathcal{S}(\mathcal{I})$ and a $\mathcal{J}$-density point of $B \in \mathcal{S}(\mathcal{J})$, then $\langle 0,0\rangle$ is an $\mathcal{I}\times\mathcal{J}$-density point of $A \times B$.*

PROOF. Obviously, $A \times B \in \mathcal{S}(\mathcal{I}\times\mathcal{J})$. Let $\{n_i\}$ be an arbitrary increasing sequence of positive integers. Since 0 is an $\mathcal{I}$-density point of A, there is a subsequence $\{n_{i_k}\}$ such that

$$C = \limsup_{k\to\infty}((-1,1)\backslash n_{i_k}A) \in \mathcal{I}.$$

Let $x \in (-1,1)\backslash C$. Then there exists an integer k_* such that $x \in n_{i_k}A$ for each $k \geq k_*$. Since 0 is a $\mathcal{J}$-density point of B, there is a subsequence $\{n_{i_{k_j}}\}$ such that

$$D = \limsup_{j\to\infty}((-1,1)\backslash n_{i_{k_j}}B) \in \mathcal{J}.$$

Since $\frac{x}{n_{i_{k_j}}} \in A$ for $k_j \geq k_*$, we get

$$\limsup_{j\to\infty}((-1,1)\backslash n_{i_{k_j}}(A\times B)_{x/n_{i_{k_j}}}) = D \in \mathcal{J}$$

which yields the assertion. $\square$

From Propositions 2.3 and 2.4 we derive

Corollary 2.5 *$\langle 0,0\rangle$ is an $\mathcal{I}\times\mathcal{J}$-density point of $A\times B$, where $A \in \mathcal{S}(\mathcal{I})$ and $B \in \mathcal{S}(\mathcal{J})$, if and only if 0 is an $\mathcal{I}$-density point of A and a $\mathcal{J}$-density point of B.* $\square$

If the σ-ideals $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{I}\times\mathcal{J}$ generate the respective topologies $\tau_{\mathcal{I}}$, $\tau_{\mathcal{J}}$ and $\tau_{\mathcal{I}\times\mathcal{J}}$ then, by Proposition 2.4, we easily get $\tau_{\mathcal{I}}\otimes\tau_{\mathcal{J}} \subseteq \tau_{\mathcal{I}\times\mathcal{J}}$ where $\tau_{\mathcal{I}}\otimes\tau_{\mathcal{J}}$ stands for the respective product topology. The equality cannot hold, which is shown in the following

Example 2.6 *Let $\alpha > 0$ and*

$$U = \{\langle x,y\rangle \in \mathbb{R}^2 : |y| < |x|^\alpha\} \cup \{\langle 0,0\rangle\},$$
$$V = \{\langle x,y\rangle \in \mathbb{R}^2 : |y| > |x|^\alpha\} \cup \{\langle 0,0\rangle\}.$$

Then $A \times B \not\subseteq U$ for any $A \in \tau_{\mathcal{I}}$ and $B \in \tau_{\mathcal{J}}$ such that $0 \in A \cap B$. Indeed, if $y \in B \backslash \{0\}$, we have $\langle 0, y \rangle \in (A \times B) \backslash U$. Similarly, $A \times B \not\subseteq V$ for any $A \in \tau_{\mathcal{I}}$ and $B \in \tau_{\mathcal{J}}$. However, for certain values of α, we get $U \in \tau_{\mathcal{I} \times \mathcal{J}}$ or $V \in \tau_{\mathcal{I} \times \mathcal{J}}$ (in fact, it is equivalent to $\langle 0, 0 \rangle \in \varphi_{\mathcal{I} \times \mathcal{J}}(U)$ or to $\langle 0, 0 \rangle \in \varphi_{\mathcal{I} \times \mathcal{J}}(V)$). Consider three cases:

1^0 $\alpha < 1$; then $\langle 0, 0 \rangle$ is an $\mathcal{I} \times \mathcal{J}$-density point of U, and thus, an $\mathcal{I} \times \mathcal{J}$-dispersion point of V. Indeed, take an increasing sequence $\{n_i\}$ of positive integers and let $x \neq 0$. Thus

$$n_i U_{x/n_i} = (-n_i^{1-\alpha} x^\alpha,\ n_i^{1-\alpha} x^\alpha)$$

and, consequently,

$$\limsup_{i \to \infty} ((-1, 1) \backslash n_i U_{x/n_i}) = \emptyset \in \mathcal{J}.$$

Hence $\langle 0, 0 \rangle \in \varphi_{\mathcal{I} \times \mathcal{J}}(U)$.

2^0 $\alpha = 1$; then $\langle 0, 0 \rangle$ is neither an $\mathcal{I} \times \mathcal{J}$-density point of U nor an $\mathcal{I} \times \mathcal{J}$-density point of V. Indeed, take an increasing sequence $\{n_i\}$ of positive integers and let $A \in \mathcal{I}$. Choose $x \in (0, 1) \backslash A$. Then

$$\limsup_{i \to \infty} ((-1, 1) \backslash n_i U_{x/n_i}) = (-1, -x] \cup [x, 1) \notin \mathcal{J}$$

and

$$\limsup_{i \to \infty} ((-1, 1) \backslash n_i U_{x/n_i}) = [-x, x] \notin \mathcal{J},$$

which implies that $\langle 0, 0 \rangle \notin \varphi_{\mathcal{I} \times \mathcal{J}}(U)$ and $\langle 0, 0 \rangle \notin \varphi_{\mathcal{I} \times \mathcal{J}}(V)$.

3^0 $\alpha > 1$; then $\langle 0, 0 \rangle$ is an $\mathcal{I} \times \mathcal{J}$-density point of V, and thus, an $\mathcal{I} \times \mathcal{J}$-dispersion point of U. The proof is similar to that in 1^0.

3 $\mathbb{L} \times \mathbb{K}$-density points

It is known that the pair $\langle \mathcal{S}(\mathbb{L} \times \mathbb{K}), (\mathbb{L} \times \mathbb{K}) \rangle$ fulfils ccc (see [G, Th.2.3], [F1, Prop.8G(a)]). So, it is worth to verify whether the corresponding operator $\varphi_{\mathbb{L} \times \mathbb{K}}$ satisfies condition (4) stated in Section 1. Instead of $\varphi_{\mathbb{L} \times \mathbb{K}}$ we shall write in short φ. The symbol $A \sim B$ (see Section 1) will be reserved for the case $A, B \subseteq \mathbb{R}^2$ and $A \triangle B \in \mathbb{L} \times \mathbb{K}$.

Lemma 3.1 *([F2], [B, Prop.2.1]). For each $A \in \mathcal{S}(\mathbb{L} \times \mathbb{K})$ there are sequences of open sets U_n in $\mathbb{R}$ and of Borel sets B_n in $\mathbb{R}$ such that $A \sim \bigcup_{n \in \mathbb{N}} B_n \times U_n$.*

Corollary 3.2 *For any set $A \in \mathcal{S}(\mathbb{L} \times \mathbb{K}) \setminus (\mathbb{L} \times \mathbb{K})$ there are sequences of nonempty open sets U_n in $\mathbb{R}$ and of Borel sets $E_n \notin \mathbb{L}$, such that $E_n = \varphi_{\mathbb{L}}(E_n)$ and $A \sim \bigcup_{n \in \mathbb{N}} E_n \times U_n$.*

PROOF. Choose sets B_n and U_n according to Lemma 3.1. Let $M = \{n \in \mathbb{N} : B_n \notin \mathbb{L}$ and $U_n \neq \emptyset\}$. Let $E_n = \varphi_{\mathbb{L}}(B_n)$ for $n \in M$. Then $E_n = \varphi_{\mathbb{L}}(E_n)$ and

$$A \sim \bigcup_{n \in \mathbb{N}} B_n \times U_n \sim \bigcup_{n \in M} B_n \times U_n \sim \bigcup_{n \in M} E_n \times U_n. \quad \square$$

Proposition 3.3 *For any $A \in \mathcal{S}(\mathbb{L} \times \mathbb{K})$ we have $A \sim \varphi(A)$.*

PROOF. Let $\mathcal{F}$ be a family of sets of the form $B = \bigcup_{n \in \mathbb{N}} E_n \times U_n$ where E_n are Borel, $\varphi_{\mathbb{L}}(E_n)$, and U_n are open and nonempty. By Corollary 3.2 and Lemma 1.2, it suffices to show that $B \subseteq \varphi(B)$ for any $B \in \mathcal{F}$. So, let B be as above and let $\langle x_0, y_0 \rangle \in B$. We may assume that $\langle x_0, y_0 \rangle = \langle 0, 0 \rangle$. Pick $m \in \mathbb{N}$ such that $\langle 0, 0 \rangle \in E_m \times U_m$. Since $0 \in \varphi_{\mathbb{L}}(E_m)$ and $0 \in \varphi_{\mathbb{K}}(U_m)$, by Proposition 2.4, we get $\langle 0, 0 \rangle \in \varphi(E_m \times U_m)$. Then obviously $\langle 0, 0 \rangle \in \varphi(B)$. $\square$

Corollary 3.4 *$\tau_{\mathbb{L} \times \mathbb{K}}$ forms a topology.* $\square$

4 $\mathbb{K} \times \mathbb{L}$-density points

We want to get the respective topology $\tau_{\mathbb{K} \times \mathbb{L}}$. Since $\langle \mathcal{S}(\mathbb{K} \times \mathbb{L}), (\mathbb{K} \times \mathbb{L}) \rangle$ fulfils ccc (see [G, Th. 2.3], [F1, Prop.8G(a)]), it suffices to verify condition (4) for $\varphi_{\mathbb{K} \times \mathbb{L}}$. Here the symbol $\sim$ is associated with $\mathbb{K} \times \mathbb{L}$. In this case the following approximation lemma works.

Lemma 4.1 ([B, Prop.2.4], [B1]) *For each $A \in \mathcal{S}(\mathbb{K} \times \mathbb{L})$, there exists a set $B \subseteq \mathbb{R}^2$ of type G_δ such that $A \sim B$.* $\square$

The proof of (4) for $\mathbb{K} \times \mathbb{L}$ is more complicated than that for $\mathbb{L} \times \mathbb{K}$. A careful choice of subsequences and, if necessary, the consideration of the diagonal sequence will serve as the basic tools.

The Lebesgue measure on $\mathbb{R}$ will be denoted by λ.

Lemma 4.2 *Let $B \in \mathcal{B}(\mathbb{R}^2)$ and $x \in \mathbb{R}$. Then*

(a)

$$0 \in \varphi_{\mathbb{L}}(B_x) \iff \forall_{\{n_i\}} \exists_{\{n_{i_k}\}} \lim_{j \to \infty} \lambda\Big(\bigcap_{k=j}^{\infty} (n_{i_k} B_x) \cap (-1, 1)\Big) = 2,$$

(b)

$$0 \in (\varphi_{\mathbb{K} \times \mathbb{L}}(B))_x \iff \forall_{\{n_i\}} \exists_{\{n_{i_k}\}} \exists_{D \in \mathbb{K}} \forall_{t \in (-1,1) \setminus D}$$

$$\lim_{j \to \infty} \lambda\Big(\bigcap_{k=j}^{\infty} (n_{i_k} B_{x + (t/n_{i_k})}) \cap (-1, 1)\Big) = 2.$$

PROOF. (a) Since

$$\limsup_{k\to\infty}((-1,1)\backslash n_{i_k}B_x) \in \mathbb{L} \iff \lambda((\liminf_{k\to\infty} n_{i_k}B_x)\cap(-1,1)) = 2$$

$$\iff \lim_{j\to\infty}\lambda(\bigcap_{k=j}^{\infty}(n_{i_k}B_x)\cap(-1,1)) = 2,$$

we get the assertion immediately , by the definition of an $\mathbb{L}$-density point. The proof of (b) is analogous. □

Proposition 4.3 *$A \sim \varphi_{\mathbb{K}\times\mathbb{L}}(A)$ for each $A \in \mathcal{S}(\mathbb{K}\times\mathbb{L})$.*

PROOF. For any set $B \subseteq \mathbb{R}^2$ of type G_δ, we shall find $E \in \mathbb{K}$ such that

$$(*) \qquad \varphi_{\mathbb{L}}(B_x) \subseteq (\varphi_{\mathbb{K}\times\mathbb{L}}(B))_x$$

for each $x \notin E$. We then have

$$\forall_{x\notin E} \quad B_x\backslash(\varphi_{\mathbb{K}\times\mathbb{L}}(B))_x \subseteq B_x\backslash\varphi_{\mathbb{L}}(B_x).$$

But $B_x\backslash\varphi_{\mathbb{L}}(B_x) \in \mathbb{L}$ by the Lebesgue density theorem and thus, $B\backslash\varphi_{\mathbb{K}\times\mathbb{L}}(B) \in \mathbb{K}\times\mathbb{L}$. So, by Lemmas 4.1 and 1.2, that will complete the proof.

Assume that $B = \bigcap_{r\in\mathbb{N}} B_r$ where B_r is open and $B_{r+1} \subseteq B_r$ for each $r \in \mathbb{N}$. Let $\{U_n\}$ be a sequence of open nonempty sets that form a basis of the topology in $\mathbb{R}$. For any $r, n \in \mathbb{N}$, let

$$E_n^r = \{x \in \mathbb{R} : U_n \subseteq (B_r)_x\}.$$

Observe that

$$\mathbb{R}\backslash E_n^r = pr_1((\mathbb{R}\times U_n)\cap(\mathbb{R}^2\backslash B_r)).$$

Therefore $\mathbb{R}\backslash E_n^r$ is of type F_σ (cf. [K, §20, V]) and E_n^r - of type G_δ. Hence E_n^r has the Baire property. Let $H_n^r = \varphi_{\mathbb{K}}(E_n^r)\cap E_n^r$. We have $H_n^r \subseteq E_n^r$ and $E_n^r\backslash H_n^r \in \mathbb{K}$, $\varphi_{\mathbb{K}}(H_n^r) = H_n^r$, by the properties of $\varphi_{\mathbb{K}}$. Let

$$E = \bigcup_{r,n\in\mathbb{N}} (E_n^r\backslash H_n^r).$$

Obviously, $E \in \mathbb{K}$.

To show $(*)$, fix $x \notin E$ and $y \in \varphi_{\mathbb{L}}(B_x)$. For simplicity, assume that $y = 0$. To prove that $0 \in (\varphi_{\mathbb{K}\times\mathbb{L}}(B))_x$, we shall use the statement (b) of Lemma 4.2. Consider an increasing sequence $\{n_i\}$ of positive integers. Since $0 \in \varphi_{\mathbb{L}}(B_x)$, we choose (by Lemma 4.2 (a)) a subsequence $\{n_{i_k}\}$ of $\{n_i\}$ such that $\lim_{j\to\infty}\lambda(\bigcap_{k=j}^{\infty}(n_{i_k}B_x)\cap(-1,1))=2$. Hence, from $B\subseteq B_r$ for every r, we deduce that

$$(**)\qquad \forall_{r\in\mathbb{N}}\ \lim_{j\to\infty}\lambda(\bigcap_{k=j}^{\infty}(n_{i_k}(B_r)_x)\cap(-1,1))=2.$$

Note that $(**)$ remains true if we replace $\{n_{i_k}\}$ by a subsequence.
Claim. There are a subsequence $\{m_k\}$ of $\{n_{i_k}\}$ and a set $D_1\in\mathbb{K}$ such that

$$\forall_{p\in\mathbb{N}}\forall_{t\in(-1,1)\setminus D_1}\ \lambda(\bigcap_{j=p}^{\infty}(m_j(B_1)_{x+(t/m_j)})\cap(-1,1))\geq 2-\frac{1}{2^{p-1}}.$$

PROOF. From $(**)$ it follows that, for each $p\in\mathbb{N}$ there is $j_p\in\mathbb{N}$ such that

$$\lambda(\bigcap_{j=j_p}^{\infty}(n_{i_j}(B_1)_x\cap(-1,1))\geq 2-\frac{1}{2^p}.$$

We may assume that $j_{p+1}>j_p$ for every p. Then

$$(\mathrm{I})\qquad \forall_{p\in\mathbb{N}}\ \lambda(\bigcap_{k=p}^{\infty}(n_{i_{j_k}}(B_1)_x\cap(-1,1))\geq 2-\frac{1}{2^p}.$$

Put $l_k=n_{i_{j_k}}$ for $k\in\mathbb{N}$. Since $(B_1)_x$ is open, there is a sequence $\{U_{s_i}\}$ of sets from the basis, such that $(B_1)_x=\bigcup_{i=1}^{\infty}U_{s_i}$. Fix any $p\in\mathbb{N}$. By (I), we can choose $i(p)\in\mathbb{N}$ such that

$$(\mathrm{II})\qquad \lambda(\bigcap_{k=p}^{\infty}(l_k\bigcup_{i=1}^{i(p)}U_{s_i})\cap(-1,1))\geq 2-\frac{1}{2^{p-1}}.$$

Observe that (II) remains true, if we replace $\{l_k\}$ by a subsequence. Since $\bigcup_{i=1}^{i(p)}U_{s_i}\subseteq(B_1)_x$, therefore from the definition of the sets E_n^1 we get $x\in\bigcap_{i=1}^{i(p)}E_{s_i}^1$. Put $H_p=\bigcap_{i=1}^{i(p)}H_{s_i}^1$. From $x\notin E$ it follows that $x\in H_p$. By the properties of $\varphi_{\mathbb{K}}$ and the sets H_n^1 we have $x\in\varphi_{\mathbb{K}}(H_p)$.

Now, for each $p\in\mathbb{N}$ we will define inductively a sequence $\{m_k^{(p)}\}$ such that

$$(\mathrm{III})\qquad D_p^1:=\limsup_{k\to\infty}((-1,1)\setminus m_k^{(p)}(H_p-x))\in\mathbb{K}$$

and

$$(\mathrm{IV})\qquad \forall_{t\in(-1,1)\setminus D_p^1}\ \lambda(\bigcap_{k=p}^{\infty}(m_k^{(p)}(B_1)_{x+(t/m_i^{(p)})})\cap(-1,1))\geq 2-\frac{1}{2^{p-1}}.$$

So, let $p = 1$. Since $x \in \varphi_{\mathbb{K}}(H_1)$, there is a subsequence $\{m_k^{(1)}\}$ of $\{l_k\}$ such that (III) holds. If $t \in (-1,1)\backslash D_1^1$, we get $x + \frac{t}{m_k^{(1)}} \in H_1$ for almost all k. We may assume that this holds for all k. Thus

$$\forall_{k\in\mathbb{N}} \ (B_1)_{x+(t/m_k^{(1)})} \supseteq \bigcup_{i=1}^{i(1)} U_{s_i}$$

and, by (II) we get (IV).

Assume that for $p \in \mathbb{N}$ we have chosen a sequence $\{m_i^{(p)}\}$ fulfilling (III) and (IV). In the way indicated above we select a subsequence $\{m_i^{(p+1)}\}$ of $\{m_i^{(p)}\}$ such that conditions (III) and (IV) with p replaced by $p+1$ hold. The induction is finished.

If we put $D_1 = \bigcup_{p=1}^{\infty} D_p^1$ and $m_k = m_k^{(k)}$ for $k \in \mathbb{N}$, we obtain the assertion of Claim. □

Next we use induction with respect to r. Let $k_j^{(1)} = m_j$ for $j \in \mathbb{N}$. Assume that for $r \in \mathbb{N}$ we have chosen a sequence $\{k_j^{(r)}\}$ and a set $D_r \in \mathbb{K}$ such that

$$\text{(V)} \quad \forall_{p\in\mathbb{N}}\forall_{t\in(-1,1)\backslash D_r} \ \lambda\Big(\bigcap_{j=p}^{\infty}(k_j^{(r)}(B_r)_{x+(t/k_j^{(r)})}) \cap (-1,1)\Big) \geq 2 - \frac{1}{2^{p-1}}.$$

If we repeat the procedure described in Claim, we obtain a subsequence $\{k_j^{(r+1)}\}$ of $\{k_j^{(r)}\}$ and a set $D_{r+1} \in \mathbb{K}$ such that condition (V) with r replaced by $r+1$ holds. Put $k_j = k_i^{(j)}$ for $j \in \mathbb{N}$ and $D = \bigcup_{r=1}^{\infty} D_r$. We infer that $D \in \mathbb{K}$ and

$$\forall_{p,r\in\mathbb{N}}\forall_{t\in(-1,1)\backslash D} \ \lambda\Big(\bigcap_{i=p}^{\infty}(k_i(B_r)_{x+(t/k_i)}) \cap (-1,1)\Big) \geq 2 - \frac{1}{2^{p-1}}.$$

Consider $r \to \infty$ and $p \to \infty$. Then, by Lemma 4.2 (b) we get $0 \in (\varphi_{\mathbb{K}\times\mathbb{L}}(B))_x$. That gives $(*)$ as desired. □

Corollary 4.4 $\tau_{\mathbb{K}\times\mathbb{L}}$ *forms a topology.* □

5 Concluding remarks

One can think that there is a strict dependence between our main results, Propositions 3.3 and 4.3; for instance, maybe at least one of them follows from the other. That is an open problem. Our proofs of those two facts present rather different techniques, so a simple connection is not visible. In this section we want to give some more information on relationships between $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$.

From the well-known expression $\mathbb{R} = A \cup B$ where $A \in \mathbb{K}$ and $B \in \mathbb{L}$ ([O, Th.1.6]) we get $\mathbb{R}^2 = (A \times \mathbb{R}) \cup (B \times \mathbb{R})$ and thus it follows that $A \times \mathbb{R} \in \mathbb{K} \times \mathbb{L}$ and $B \times \mathbb{R} \in \mathbb{L} \times \mathbb{K}$. Hence there is no inclusion between $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ ([M, Th.1.2]). However, this does not exclude the existence of a Borel isomorphism between $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ (one can even conjecture that the mapping $\langle x, y \rangle \mapsto \langle y, x \rangle$ is good). We will give some comments concerning that problem.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals of subsets of X, we say that a bijection $f : X \to X$ is an *isomorphism* between $\mathcal{I}$ and $\mathcal{J}$ if, for the mapping $f^* : \mathcal{P}(X) \to \mathcal{P}(X)$ given by $f^*(E) = f[E]$ (the image of f), we have $f^*[\mathcal{I}] = \mathcal{J}$; if f is Borel measurable, it is called a *Borel isomorphism.* Note that an expression of type $X = C \cup D$ for $C \in \mathcal{I}$ and $D \in \mathcal{J}$, when CH and some natural properties of σ-ideals $\mathcal{I}$ and $\mathcal{J}$ are supposed, leads us to an isomorphism f between $\mathcal{I}$ and $\mathcal{J}$. Indeed, here a scheme used in the classical Sierpiński-Erdös theorem holds. (See [O,Th.19.5]. Moreover, one can ensure that $f = f^{-1}$ which guarantees $f^*[\mathcal{I}] = \mathcal{J}$ and $f^*[\mathcal{J}] = \mathcal{I}$ simultaneously.) Since the Sierpiński-Erdös theorem works for $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ (see [M]), we get, under CH, an isomorphism between those products. This proof, however, does not give a Borel isomorphism. Moreover, in another model of ZFC, any isomorphism between $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ is excluded. Indeed, if, for an ideal $\mathcal{I}$ of subsets of X, one defines

$$\mathrm{add}(\mathcal{I}) = min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \& \bigcup \mathcal{F} \notin \mathcal{I}\}$$

then $\mathrm{add}(\mathbb{K} \times \mathbb{L}) = \omega_1$ and $\mathrm{add}(\mathbb{L} \times \mathbb{K}) = \mathrm{add}(\mathbb{L})$ ([CP] and [F2]). Since $\mathrm{add}(\mathbb{L}) > \omega_1$ is consistent, so is $\mathrm{add}(\mathbb{L} \times \mathbb{K}) > \mathrm{add}(\mathbb{K} \times \mathbb{L})$. Thus, in any model of ZFC in which $\mathrm{add}(\mathbb{L}) > \omega_1$, there is no isomorphism between $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$. Consequently, there is no Borel isomorphism. But, it turns out that this last statement is true in ZFC. Namely, Gavalec in [G] posed the following question: Are the Boolean algebras $\mathcal{B}(\mathbb{R}^2)/(\mathbb{K} \times \mathbb{L})$ and $\mathcal{B}(\mathbb{R}^2)/(\mathbb{L} \times \mathbb{K})$ isomorphic? In the final remark of [G] we read that J. Truss excluded such an isomorphism. Observe that if there were a Borel isomorphism between $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$, the answer to Gavalec's question would be positive. So, the result of Truss implies that $\mathbb{K} \times \mathbb{L}$ and $\mathbb{L} \times \mathbb{K}$ are not Borel isomorphic (in ZFC). The following important question raised by the referee remains unsolved: Are $\tau_{\mathbb{K} \times \mathbb{L}}$ and $\tau_{\mathbb{L} \times \mathbb{K}}$ homeomorphic? We conjecture that the answer is "no". It would be interesting to study the argument of Truss since the above problem seems related to Gavalec's question. The original proof in [T] uses esentially the technique of forcing. A version translating it into the language of Boolean algebras has been sent us lately by Fremlin [F3].

Notice that there is no inclusion between our topologies $\tau_{\mathbb{K} \times \mathbb{L}}$ and $\tau_{\mathbb{L} \times \mathbb{K}}$.

Indeed, if A and B are as above, then

$$\mathbb{R} \times A \in \tau_{\mathbb{K}\times\mathbb{L}} \setminus \tau_{\mathbb{L}\times\mathbb{K}} \quad \text{and} \quad \mathbb{R} \times B \in \tau_{\mathbb{L}\times\mathbb{K}} \setminus \tau_{\mathbb{K}\times\mathbb{L}}.$$

Moreover, A. Miller (oral communication) has observed that, if $g(x,y) = x+y$, then

$$g^{-1}[A] \in \tau_{\mathbb{K}\times\mathbb{L}} \setminus \tau_{\mathbb{L}\times\mathbb{K}} \quad \text{and} \quad g^{-1}[B] \in \tau_{\mathbb{L}\times\mathbb{K}} \setminus \tau_{\mathbb{K}\times\mathbb{L}}$$

Here $g^{-1}[A]$ and $g^{-1}[B]$ are invariant with respect to $\langle x,y\rangle \mapsto \langle y,x\rangle$ which therefore cannot be a homeomorphism between $\tau_{\mathbb{K}\times\mathbb{L}}$ and $\tau_{\mathbb{L}\times\mathbb{K}}$.

Finally, let us mention some simple properties of $\tau_{\mathbb{K}\times\mathbb{L}}$ and $\tau_{\mathbb{L}\times\mathbb{K}}$ and list some questions. The proofs are omitted since they are similar to those for the classical density topology and the Wilczyński $\mathcal{I}$-density topology. Besides the analogues, several problems appear; they can make material for further investigations.

From now on, assume that $\mathcal{J} = \mathbb{K} \times \mathbb{L}$ or $\mathcal{J} = \mathbb{L} \times \mathbb{K}$.

1. Since $\tau_{\mathcal{J}}$ is finer than the natural topology on $\mathbb{R}^2$, it must be Hausdorff. Is it regular? (Note: K. Ciesielski has informed us that $\tau_{\mathbb{L}\times\mathbb{K}}$ is not regular but his argument does not work for $\tau_{\mathbb{K}\times\mathbb{L}}$).

2. A set E is closed and discrete in $\tau_{\mathcal{J}}$ iff $E \in \mathcal{J}$.

3. A set E is compact in $\tau_{\mathcal{J}}$ iff it is finite.

4. $\tau_{\mathcal{J}}$ is neither separable nor has the Lindelöf property.
   (For the proofs of (2)–(4), see [CLO, Th.2.6.2].)

5. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called *$\mathcal{J}$-approximately continuous at* $p \in \mathbb{R}^2$ if

   $$p \in \varphi_{\mathcal{J}}(f^{-1}[(f(p) - \varepsilon,\, f(p) + \varepsilon)])$$

   for each $\varepsilon > 0$. We say that f is *$\mathcal{J}$-approximately continuous* if $f^{-1}[(a,b)] \in \tau_{\mathcal{J}}$ for every interval (a,b). Obviously, f is $\mathcal{J}$-approximately continuous iff it is is $\mathcal{J}$-approximately continuous at every point. Observe that (cf. Example 2.6) $f : \mathbb{R}^2 \to \mathbb{R}$ given by

   $$f(x,y) = \begin{cases} \frac{y^2}{|x|+y^2} & \text{for } \langle x,y\rangle \neq \langle 0,0\rangle \\ 0 & \text{for } \langle x,y\rangle = \langle 0,0\rangle \end{cases}$$

   is $\mathcal{J}$-approximately continuous and discontinuous (in the usual sense) at $\langle 0,0\rangle$.

6. The class of all bounded $\mathcal{J}$-approximate continuous functions with the sup-norm forms a Banach space (cf. [CLO, Th. 2.5.5]).

7. Each function $\mathcal{J}$-approximately continuous $\mathcal{J}$-almost everywhere on $\mathbb{R}^2$ must be $\mathcal{S}(\mathcal{J})$-measurable (cf. [PWW, Th. 6]). Is the converse true?

8. Does every $\mathcal{J}$-approximately continuous function belong to Baire class 1? (Compare [CLO, Th.1.3.1 and 2.5.5].)

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