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ON A GENERALIZED DOMINATED CONVERGENCE THEOREM FOR THE AP INTEGRAL

Abstract

We prove a Harnack extension and a form of the generalized dominated convergence theorem for the AP integral.

1 Introduction

The AP integral, or more precisely, the approximately continuous Perron integral was defined in Burkill [1] and its Riemann-type definition was given in [2]. As was pointed out by Liao [4], [8], the property ACG^*_{ap} in [2] doesn't describe the primitive of an AP integrable function. A correct version was obtained in [4]. In this note we shall give a Harnack extension and a form of the generalized dominated convergence theorem for the AP integral.

Throughout all functions and sets will be assumed to be Lebesgue measurable. We recall the following definitions and results.

approximate neighborhood: An approximate neighborhood of $x \in [a,b]$ is a measurable set $D_x \subset [a,b]$ containing x and having density 1 at x.

AFC of E: Let $E \subset [a,b]$ and for every $x \in E$ choose an approximate neighborhood, $S_x \subset [a,b]$, of x. Put $S = \{S_x : x \in E\}$ and let $[u,v]$ be a subinterval of $[a,b]$. If there exists an $x \in E \cap [u,v]$ such that $u, v \in S_x$, we call x an associated point of $[u,v]$. The set of all intervals having an associated point $x \in E$ is called an approximate full cover, or AFC, of E. An AFC will be denoted by Δ.

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Δ-partial division on E: If Δ is an AFC of $E \subset [a,b]$, we call a finite set of nonoverlapping intervals $[u_i, v_i] \in \Delta$, $i = 1, 2, \ldots, n$ together with associated points x_i a Δ-partial division on E, denoted by $\{([u_i, v_i]; x_i) : i = 1, 2, \ldots, n\}$ or simply $\{([u_i, v_i]; x_i)\}$.

Δ-division on E: If $\bigcup_{i=1}^{n} [u_i, v_i] \supset E$, we call $\{([u_i, v_i]; x_i) : i = 1, 2, \ldots, n\}$ a Δ-division on E.

ASL: We say that a function F satisfies the approximate strong Lusin condition, or simply ASL, on a subset X of $[a,b]$ if for every set E of measure zero and for every $\epsilon > 0$ there exists an AFC Δ of X such that for any Δ partial-division D on $E \cap X$ we have $(D)\sum |F(u,v)| < \epsilon$ where $F(u,v) = F(v) - F(u)$.

UASL: A sequence $\{F_n\}$ of functions is said to satisfy the uniformly approximately strong Lusin condition, or $\{F_n\} \in UASL$ for short, if the AFC Δ in the definition of ASL is independent of n.

$\mathbf{ACG^*_{ap}}$: A function F is said to satisfy ACG^*_{ap} on E, or $F \in ACG^*_{ap}(E)$, if there exists a sequence $\{X_n\}$ such that $E = \bigcup_{k=1}^{\infty} X_k$ and $F \in AC^*_{ap}(X_k)$ for each k, i.e. for every $\epsilon > 0$ there exist an AFC Δ and $\eta > 0$ such that for any Δ-partial division $\{([u_i, v_i]; x_i) : i = 1, 2, \ldots, n\}$ on X_k, whenever $\sum_{i=1}^{k} (v_i - u_i) < \eta$, we have $\sum_{i=1}^{k} |F(v_i) - F(u_i)| < \epsilon$.

$\mathbf{UACG^*_{ap}}$: We say $\{F_n\} \in UACG^*_{ap}(E)$ if for each k the AFC Δ and $\eta > 0$ in the definition of $AC^*_{ap}(X_k)$ are independent of n.

ACG: A function F is said to be ACG on E, or $F \in ACG(E)$, if $E = \bigcup_{k=1}^{\infty} X_k$ such that $F \in AC(X_k)$ for each k, i.e. for every $\epsilon > 0$ there exits $\eta > 0$ such that whenever $\{[u,v]\}$ is a finite or infinite sequence of nonoverlapping intervals with $u, v \in X_k$ satisfying $\sum(v-u) < \eta$ we have $\sum |F(u,v)| < \epsilon$.

(ACG): A function is said to be (ACG) on E, or $F \in (ACG)(E)$, if $F \in ACG(E)$ and each X_k in the definition of $ACG(E)$ is closed.

UACG(E): A sequence $\{F_n\}$ of functions is said to be UACG on E, or $\{F_n\} \in UACG(E)$, if $E = \bigcup_{k=1}^{\infty} X_k$ with $\{F_n\} \in UAC(X_k)$ for each k, i.e. the $\eta > 0$ in the definition of $AC(X_k)$ are independent of n.

A function f is said to be AP integrable to A on $[a,b]$ if for every $\epsilon > 0$ there is an AFC Δ of $[a,b]$ such that for every Δ-division $D = \{([u,v];\xi)\}$ of $[a,b]$ we have $|(D)\sum f(\xi)(v-u) - A| < \epsilon$. We write $A = \int_a^b f$.

We also recall that the Henstock lemma holds for the AP integral: If f is AP integrable on $[a,b]$ with primitive F, then for every $\epsilon > 0$ there is an AFC Δ of $[a,b]$ such that for every Δ-partial division $\{([u_i,v_i];\xi)\}$ on $[a,b]$ we have $(D)\sum |f(\xi)(v-u) - F(u,v)| < \epsilon$, where $F(u,v) = F(v) - F(u)$.

In [4, Proposition 3.12] Liao and Chew proved that if f is AP integrable on $[a,b]$ with primitive F, then F satisfies ASL on $[a,b]$.

For more details on the above definitions and results, see [2], [3, Section 22], [4], [5], [6, Chapter 7.8], [7], [8], [9]. In what follows we write $(D)\sum_P$ to denote the sum over D such that condition P holds.

$|S|$ will denote the Lebesgue measure of a Lebesgue measurable set S.

2 Some Results

Theorem 2.1 (Harnack extension) *Let X be a closed set in $[a,b]$ with H the set of all points of density in X. Set $(a,b)\setminus X = \bigcup_{k=1}^{\infty}(a_k,b_k)$. Suppose the following conditions are satisfied.*

(i) f is AP integrable on X (i.e. $f\chi_X$ is AP integrable on $[a,b]$, where χ_X is the characteristic function of X) and on each of the intervals $[a_k,b_k]$.

(ii) The series $\sum_{k=1}^{\infty}\left|\int_{a_k}^{b_k} f\right|$ converges.

(iii) For given $\epsilon > 0$ there exists an AFC Δ of $X\setminus H$ such that for any Δ-partial division $D = \{([u,v];\xi)\}$ on $X\setminus H$ we have

$$(D)\sum_{v\in(a_k,b_k)}\left|\int_{a_k}^{v} f\right| < \epsilon \quad \text{and} \quad (D)\sum_{u\in(a_k,b_k)}\left|\int_{u}^{b_k} f\right| < \epsilon.$$

Then f is AP integrable on $[a,b]$ with $\int_a^b f = \int_a^b f\chi_X + \sum_{k=1}^{\infty}\int_{a_k}^{b_k} f$.

PROOF. For simplicity we may suppose that $f(a_k) = f(b_k) = 0$ for all k. By condition (i) f is AP integrable on $[a_k,b_k]$ for each k. Given $\epsilon > 0$ there exists an AFC Δ_k of $[a_k,b_k]$ such that for any Δ_k partial-division $D_k = \{([u,v];\xi)\}$ on $[a_k,b_k]$ we have

$$(D_k)\sum\left|f(\xi)(v-u) - \int_u^v f\right| < \frac{\epsilon}{2^k} \tag{1}$$

We may suppose that for each $([u,v];\xi) \in \Delta_k$ we have

(I) $[u,v] \subset (a_k, b_k)$ when $\xi \in (a_k, b_k)$

(II) $(\xi, v] \subset (a_k, b_k)$ when $\xi = a_k$

(III) $[u, \xi) \subset (a_k, b_k)$ when $\xi = b_k$.

Now define an AFC Δ of $[a,b]$ as follows:

(i)′ $\Delta = \Delta_k$ on each $[a_k, b_k]$

(ii)′ By condition (ii) we may choose a positive integer N so that $\sum\limits_{k=N+1}^{\infty} \left|\int_{a_k}^{b_k} f\right| < \epsilon$, and when $([u,v];\xi) \in \Delta$ with $\xi \in X \setminus \{a_1, b_1, \ldots, a_N, b_N\}$, we have $[u,v] \cap \left(\bigcup\limits_{k=1}^{N} [a_k, b_k]\right) = \emptyset$.

(iii)′ If $a_j \neq b_l$ for all $1 \leq j, l \leq N$ with $j \neq l$, we choose the same N as in (ii)′, and we have $[u, \xi) \cap \left(\bigcup\limits_{k=1}^{N} [a_k, b_k]\right) = \emptyset$ when $\xi = a_i$, $i = 1, 2, \ldots, N$ and $(\xi, v] \cap \left(\cup_{k=1}^{N} [a_k, b_k]\right) = \emptyset$ when $\xi = b_i$, $i = 1, 2, \ldots, N$. If $\xi = a_j = b_l$ for some $1 \leq j, l \leq N$ with $j \neq l$, then we choose $[u,v]$ so that $([u,v];\xi) \in \Delta_l$.

(iv)′ $u, v \in X$ if $\xi \in H$. This is possible since H is the set of all points of density in X.

Note that we have defined an AFC Δ of $[a,b]$ so that if $\xi \in X$ and u or $v \in (a_k, b_k)$ for some $k > N$, then $\xi \in X \setminus H$.

Let D be a Δ-division on $[a,b]$. Decompose D into $D_1, D_2, D_3, \ldots, D_N, D_X$ and D_0, where $([u,v];\xi) \in D_k$ if $\xi \in [a_k, b_k]$ for some $k = 1, 2, \ldots, N$, $([u,v];\xi) \in D_X$ if $\xi \in X \subset \left\{a_1, b_1, a_2, b_2, \ldots, a_N, b_N\right\}$ and $([u,v];\xi) \in D_0$ if $\xi \in \bigcup\limits_{k=N+1}^{\infty} (a_k, b_k)$. Then we have

$$\left|(D)\sum\left\{f(\xi) - f(\xi)\chi_X(\xi)\right\}(v-u) - \sum_{k=1}^{\infty}\int_{a_k}^{b_k} f\right|$$

$$\leq \sum_{k=1}^{N}(D_k)\sum\left|f(\xi)(v-u) - \int_u^v f\right| + (D_0)\sum\left|f(\xi)(v-u) - \int_u^v f\right|$$

$$+ (D_X)\sum_{\substack{u\in(a_k,b_k)\\k>N}}\left|\int_u^{b_k} f\right| + (D_X)\sum_{\substack{v\in(a_k,b_k)\\k>N}}\left|\int_{a_k}^{v} f\right| + (D_X)\sum_{\substack{[a_k,b_k]\subset[u,v]\\k>N}}\left|\int_{a_k}^{b_k} f\right|$$

$$< \sum_{k=1}^{N}\frac{\epsilon}{2^k} + \sum_{k=N+1}^{\infty}\frac{\epsilon}{2^k} + \epsilon + \epsilon + \sum_{k=N+1}^{\infty}\left|\int_{a_k}^{b_k} f\right| \quad \text{by (1) and (iii),}$$

$<4\epsilon$ by our choice of N.

Thus $f - f\chi_X$ is AP integrable on $[a,b]$ and so is f. □

It can be shown that Theorem 2.1 is indeed a consequence of the controlled convergence theorem given in [4, Theorem 4.2].

Definition 2.2 *Let $F : [a,b] \to \mathbb{R}$ and let X be a closed subset $[a,b]$. F is $W(X)$ if and only if for each $c, d \in X$ with $c < d$ and $(c,d) \cap X$ non-empty, given $\epsilon > 0$ there exists an AFC Δ of $(X \cap [c,d]) \setminus H'$ such that for any Δ-partial division $D = \{([u,v];\xi)\}$ on $(X \cap [c,d]) \setminus H'$ we have*

$$(D) \sum_{v \in (c_k, d_k)} |F(v) - F(c_k)| < \epsilon \text{ and } (D) \sum_{u \in (c_k, d_k)} |F(d_k) - F(u)| < \epsilon$$

where H' is the set of all points of density of $([c,d] \cap X)$ and $(c,d) \setminus X = \bigcup_{k=1}^{\infty} (c_k, d_k)$.

Theorem 2.3 *Let X be a closed subset of $[a,b]$ with $(a,b) \setminus X = \bigcup_{k=1}^{\infty} (a_k, b_k)$. Suppose that f is AP integrable on $[a,b]$ with its primitive F being $AC(X)$. Then F is $W(X)$.*

PROOF. Since $F \in AC(X)$, given $\epsilon > 0$ there exists $\eta > 0$ such that whenever $\{[u,v]\}$ is a finite or infinite sequence of non-overlapping intervals with $u, v \in X$ satisfying

$$\sum (v-u) < \eta \text{ we have } \sum |F(u,v)| < \frac{\epsilon}{2}. \tag{1}$$

By the Lebesgue density theorem (see [6]) $|X \setminus H| = 0$. So we may choose an open set G such that $G \supset X \setminus H$ and $|G| < \eta$. Since f is AP integrable on $[a,b]$, its primitive F satisfies ASL on $[a,b]$. Given $\epsilon > 0$ there exists an AFC Δ of $X \setminus H$ such that for any Δ-partial division $D = \{([u,v];\xi)\}$ on $X \setminus H$ we have

$$(D) \sum |F(u,v)| < \frac{\epsilon}{2}\,. \tag{2}$$

We may modify Δ, if necessary, so that $[u,v] \subset G$ whenever $\xi \in X \setminus H$. For any Δ-partial division $D = \{([u,v];\xi)\}$ on $X \setminus H$ we have

$$(D) \sum_{v \in (a_k, b_k)} \left| \int_{a_k}^{v} f \right| \le (D) \sum_{v \in (a_k, b_k)} \left| \int_{\xi}^{v} f \right| + (D) \sum_{v \in (a_k, b_k)} \left| \int_{\xi}^{a_k} f \right|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ by (2) and (1).}$$

Similarly, we can prove that $(D) \sum_{u \in (a_k, b_k)} \left| \int_u^{b_k} f \right| < \epsilon$. □

Corollary 2.4 **([5, Lemma 3.3])** *If f is AP integrable on $[a,b]$ with its primitive F being $AC(X)$ where X is a closed subset of $[a,b]$, then $f\chi_X$ is AP integrable on $[a,b]$ and $\int_a^b f\chi_X = \int_a^b f - \sum_{k=1}^{\infty}\int_{a_k}^{b_k} f$ where $(a,b)\setminus X = \bigcup\limits_{k=1}^{\infty}(a_k,b_k)$.*

PROOF. By Theorem 2.3 $F \in W(X)$. Put $g = f - f\chi_X$ and note that for each k, g is AP integrable on $[a_k, b_k]$ with $\int_{a_k}^{s} g = \int_{a_k}^{s} f$ for each $s \in [a_k, b_k]$. Let H be the set of all points of density of X. Then given $\epsilon > 0$ there exists an AFC Δ of $X \setminus H$ such that for any Δ-partial division $D = \{([u,v];\xi)\}$ on $X \setminus H$ we have

$$(D)\sum_{v\in(a_k,b_k)}\left|\int_{a_k}^{v} g\right| = (D)\sum_{v\in(a_k,b_k)}\left|\int_{a_k}^{v} f\right| < \epsilon$$

and

$$(D)\sum_{u\in(a_k,b_k)}\left|\int_{u}^{b_k} g\right| = (D)\sum_{u\in(a_k,b_k)}\left|\int_{u}^{b_k} f\right| < \epsilon.$$

Hence g satisfies all the conditions of Theorem 2.1. □

Corollary 2.5 *Assume the hypothesis in Corollary 2.4. Then $f\chi_X$ is Lebesgue integrable on $[a,b]$.*

PROOF. It suffices to prove that the primitive function G of $f\chi_X$ is of bounded variation on $[a,b]$. For any division $a = x_0 < x_1 < x_2 < \cdots < x_p = b$ of $[a,b]$ if $x_i \notin X$, x_i must be in (a_k, b_k) for a certain k and replacing each x_i by x_i', if necessary, we have $\int_{x_{i-1}}^{x_i} f\chi_X = \int_{x_{i-1}'}^{x_i'} f\chi_X$ with $x_i' \in X$ for $i = 1,\ldots,p$. By Corollary 2.4 $\int_{x_{i-1}'}^{x_i'} f\chi_X = \int_{x_{i-1}'}^{x_i'} f - \sum_{k=1}^{\infty} F\big([a_k,b_k]\cap[x_{i-1}',x_i']\big)$. Thus

$$\sum_{i=1}^{p}\left|\int_{x_{i-1}}^{x_i} f\chi_X\right| = \sum_{i=1}^{p}\left|\int_{x_{i-1}'}^{x_i'} f\chi_X\right| \le \sum_{i=1}^{p}\left|\int_{x_{i-1}'}^{x_i'} f\right| + \sum_{k=1}^{\infty}|F(a_k,b_k)| \le 2M$$

for some finite constant M. More precisely $M = \sup\sum\limits_{k}|F(d_k) - F(c_k)|$, where the supremum is over all finite or infinite sequence $\{[c_k,d_k]\}$ of non-overlapping intervals with $c_k, d_k \in X$ for all k. Note that M is finite because F is $AC(X)$. So F is $VB(X)$. Consequently the total variation of G on $[a,b]$ is finite. □

Lemma 2.6 **([4, Proposition 3.12])** *If F satisfies ASL in $[a,b]$ with $F'_{ap} = g$ almost everywhere in $[a,b]$, then g is AP integrable on $[a,b]$ with primitive F.*

We remark that the converse of Lemma 2.6 is valid.

Lemma 2.7 *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ with primitives $\{F_n\}$ and let X a closed subset of $[a,b]$. If $\{F_n\} \in UAC(X)$, then given $\epsilon > 0$ there exists $\eta > 0$, independent of n, such that whenever E is a measurable subset of X satisfying $|E| < \eta$ we have $\int_E |f_n| < \epsilon$ for all n.*

PROOF. The proof is similar to that of Lemma 12.2 of [7].

Lemma 2.8 *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ with primitives $\{F_n\}$ and let X be a closed subset of $[a,b]$ with $(a,b)\backslash X = \bigcup_{k=1}^{\infty}(a_k, b_k)$. Then $\{F_n\} \in UAC(X)$ if and only if the following conditions are satisfied.*

(i) For every $\epsilon > 0$ there exists $\eta > 0$, independent of n, such that if E is a measurable subset of X satisfying $|E| < \eta$, then $\int_E |f_n| < \epsilon$ for all n.

(ii) The series $\sum_{k=1}^{\infty} \left|\int_{a_k}^{b_k} f_n\right|$ converges uniformly in n.

(iii) $F_n \in W(X)$ for each n.

PROOF. ($\Longrightarrow$). This follows from Lemma 2.7, the definition of $\{F_n\} \in UAC(X)$ and Theorem 2.3.

($\Longleftarrow$). By (i) given $\epsilon > 0$ there exists $\eta > 0$, independent of n, such that whenever E is a measurable subset of X satisfying $|E| < \eta$ we have

$$\int_E |f_n| < \frac{\epsilon}{2} \quad \text{for all } n. \tag{1}$$

By (ii) there exists a positive integer $N = N(\epsilon)$ such that

$$\sum_{k=N+1}^{\infty} \left|\int_{a_k}^{b_k} f_n\right| < \frac{\epsilon}{2} \quad \text{for all } n. \tag{2}$$

Define $\delta = \frac{1}{2}\min\{\eta, b_1 - a_1, b_2 - a_2, \ldots, b_N - a_N\}$. Then for every finite or infinite sequence of non-overlapping intervals $\{[u,v]\}$ with $u, v \in X$ satisfying $\sum(v-u) < \delta$ we have

$$\begin{aligned}
&\sum |F_n(u,v)| \\
&= \sum \left| \int_u^v f_n \chi_X + \sum_{k=1}^{\infty} \int_{a_k}^{b_k} f_n \chi_{[u,v]} \right| \quad \text{by condition (iii) and Theorem 2.1} \\
&\le \sum \int_u^v |f_n \chi_X| + \sum_{k=1}^{\infty} \left| \int_{a_k}^{b_k} f_n \chi_{[u,v]} \right| \le \frac{\epsilon}{2} + \sum_{k=N+1}^{\infty} \left| \int_{a_k}^{b_k} f \right| \quad \text{by (1)}
\end{aligned}$$

since $\left|\left(\left(\cup[u,v]\right)\cap X\right)\right| = \sum\left|\left([u,v]\cap X\right)\right| < \delta$, and $(u,v)\cap(a_k,b_k)$ either contains the whole interval (a_k,b_k) or is empty with each (a_k,b_k) appearing only once. Furthermore for each $[u,v]$, $(v-u) < \min\{\eta, b_1-a_1, b_2-a_2, \ldots, b_N-a_N\}$ and $u,v \in X$. Therefore $(u,v)\cap(a_k,b_k) = \emptyset$ for $k = 1,2,\ldots,N$. Consequently by (2), $\sum|F_n(u,v)| < \epsilon$ for all n and hence $\{F_n\} \in UAC(X)$. □

3 The Generalized Dominated Convergence Theorem

In this section we shall state and prove the generalized dominated convergence theorem for the AP integral and show that it is the best possible in some sense. We need some Lemmas.

Lemma 3.1 ([5, Theorem 3.3], [9, Theorem 4.3]) *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ with primitives $\{F_n\}$. Suppose $\{f_n\}$ converges to a function f almost everywhere on $[a,b]$. The following conditions (A) and (B) are equivalent.*

(A) $\{F_n\} \in UACG^*_{ap}([a,b])$

(B) (1) $\{F_n\}$ *satisfies UASL on* $[a,b]$.

(2) For every $\epsilon > 0$ there exists a closed set $E \subseteq [a,b]$ with $|[a,b]\setminus E| < \epsilon$ such that $\{F_n\} \in UAC(E)$.

Furthermore if (A) or (B) hold, then f is AP integrable on $[a,b]$ with

$$\lim_{n\to\infty}\int_a^x f_n = \int_a^x f \quad \text{for } x \in [a,b].$$

Lemma 3.2 ([5, Lemma 5.2]) *If $\{F_n\} \in UACG(E)$, then there exist closed sets $E_1, E_2, \ldots, E_r, \ldots$ such that $E_1 \subseteq E_2 \subseteq \ldots$, $E = \bigcup_{r=1}^{\infty} E_r \cup N$ with $|N| = 0$ and $\{F_n\} \in UAC(E_r)$, $r = 1,2,\ldots$.*

Theorem 3.3 *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ with primitives $\{F_n\}$. Suppose the following conditions are satisfied.*

(i) $f_n(x) \to f(x)$ *for almost all x in $[a,b]$ as $n \to \infty$.*

(ii) $[a,b] = \bigcup_{k=1}^{\infty} X_k$ *with each X_k closed and for each k, $F_n \in W(X_k)$ for each n.*

(iii) For each $k = 1,2,\ldots$ there exists functions g_k, h_k such that g_k, h_k are each Lebesgue integrable on X_k and for almost all $x \in X_k$ we have

$$g_k(x) \le f_n(x) \le h_k(x) \quad \text{for all } n.$$

(iv) For each $k = 1, 2, \ldots$, write $(a,b) \setminus X_k = \bigcup_{j=1}^{\infty} (c_j^{(k)}, d_j^{(k)})$. Then the series $\sum_{j=1}^{\infty} \left| \int_{c_j^{(k)}}^{d_j^{(k)}} f_n \right|$ converges uniformly in n.

(v) F_n converges pointwise to a function F on $[a,b]$ and $\{F_n\}$ satisfies UASL on $[a,b]$.

Then f is AP integrable on $[a,b]$ with primitive F.

PROOF. By the Baire category theorem there is a closed interval I contained in X_j for some j. By (iii) and the dominated convergence theorem, f is Lebesgue integrable on I with primitive F and $(L)\int_I f_n \to (L)\int_I f$ as $n \to \infty$. A point x is said to be regular if there is an interval I_x containing x such that f is AP integrable on every subinterval of I_x with primitive F and we have just shown that the set of all regular points is non-empty.

Let Q be the set of all non-regular points. We claim that Q is empty. Suppose not. In view of the Cauchy extension [2, Theorem 15] for the AP integral, Q is perfect and $Q = \bigcup_{i=1}^{\infty} (Q \cap X_i)$. By the Baire category theorem again there is an interval (s,t) such that $(s,t) \cap Q = (s,t) \cap X_q$ for some q and both sets are non-empty. Let $[c,d]$ be the smallest closed interval containing $(s,t) \cap X_q$. Note that $g_q(x) \le f_n(x) \le h_q(x)$ for all n for almost all x in X_q and each g_q, h_q is Lebesgue integrable on X_q. Write $X = [c,d] \cap X_q$. By the dominated convergence theorem f is Lebesgue integrable on X and

$$(L)\int_X f_n \to (L)\int_X f \text{ as } n \to \infty \tag{1}$$

Write $(c,d) \setminus X = \bigcup_{k=1}^{\infty} (c_k, d_k)$. Note that f is AP integrable on each $[u,v] \subset (c_k, d_k)$ with primitive F. By condition (iii) for each k, given $\epsilon > 0$ there exists $\eta_k > 0$ such that whenever E is a measurable subset of X_k satisfying $|E| < \eta_k$ we have $\int_E |f_n| < \epsilon$ for all n. By conditions (ii) and (iv) and by Lemma 2.8, $\{F_n\} \in UAC(X_k)$ and so $F \in (ACG)[a,b]$. Hence $F_{ap}{}'$ exists for almost all x in $[a,b]$ (see [7, page 21]). By condition (v) F satisfies ASL on $[a,b]$. By Lemma 2.6 the function g defined on $[a,b]$ by

$$g(x) = \begin{cases} F'_{ap}(x) & \text{if } F'_{ap}(x) \text{ exists and is finite.} \\ 0, & \text{otherwise.} \end{cases}$$

is AP integrable on $[a,b]$ with primitive F. Thus F is approximately continuous on $[a,b]$. By the Cauchy extension f is AP integrable on each of the intervals $[c_k, d_k]$ and X is perfect. Since g is AP integrable on $[a,b]$ with its primitive F being $AC(X)$, by Theorem 2.3 $F \in W(X)$. Hence by Theorem 2.1 f is AP integrable on $[c,d]$ with primitive F, a contradiction. □

We shall now give the generalized dominated convergence theorem. The proof follows from Theorem 3.6 below and Lemma 3.1.

Theorem 3.4 (generalized dominated convergence theorem) *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ with primitives $\{F_n\}$. Suppose the following conditions are satisfied.*

(i) $f_n \to f$ *almost everywhere in* $[a,b]$ *as* $n \to \infty$.

(ii) $[a,b] = N \cup \bigcup_{k=1}^{\infty} X_k$, *where* $N = [a,b] \setminus \bigcup_{k=1}^{\infty} X_k$ *with each* X_k *closed,* $|N| = 0$ *and for each* k, $F_n \in W(X_k)$ *for each* n.

(iii) For each $k = 1, 2, \ldots$ *there exists functions* g_k, h_k *such that* g_k, h_k *are each Lebesgue integrable on* X_k *and for almost all* x *in* X_k *we have* $g_k(x) \le f_n(x) \le h_k(x)$ *for all* n.

(iv) For each $k = 1, 2, \ldots$, *write* $(a,b) \setminus X_k = \bigcup_{j=1}^{\infty} (c_j^{(k)}, d_j^{(k)})$. *Then the series* $\sum_{j=1}^{\infty} \left| \int_{c_j^{(k)}}^{d_j^{(k)}} f_n \right|$ *converges uniformly in* n.

(v) F_n *converges pointwise to* F *on* $[a,b]$ *with* $\{F_n\}$ *satisfies* $UASL$ *on* $[a,b]$.

Then f *is AP integrable on* $[a,b]$ *and* $\lim_{n\to\infty} \int_a^x f_n = \int_a^x f$.

Theorem 3.5 *If* $\{f_n\}$ *is a sequence of AP integrable functions on* $[a,b]$ *with primitives* $\{F_n\}$ *being* $UACG^*_{ap}([a,b])$ *and* f_n *converges almost everywhere to a function* f *on* $[a,b]$, *then there exists a subsequence* $\{f_n^{(n)}\}$ *of* $\{f_n\}$ *satisfying (i), (ii), (iii), (iv) and (v) of Theorem 3.4.*

PROOF. By Lemma 3.1 for every $\epsilon > 0$ there exists a closed set $E \subseteq [a,b]$ with $|\,[a,b] \setminus E| < \epsilon$ such that $\{F_n\} \in UAC(E)$. Hence $\{F_n\} \in UACG(Y)$ for some subset Y of $[a,b]$, where $|\,[a,b] \setminus Y| = 0$. By Lemma 3.2 there exist closed sets $X_1, X_2, \ldots, X_r, \ldots$ such that $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_r \subseteq \ldots$, $Y = \bigcup_{r=1}^{\infty} X_r \cup N_0$, where $|N_0| = 0$, and $\{F_n\} \in UAC(X_r)$, $r = 1, 2, \ldots$. By Theorem 2.3, condition (ii) of Theorem 3.4 holds. By Lemma 2.8, condition (iv) of Theorem 3.4 follows from the fact that $\{F_n\} \in UAC(X_r)$, $r = 1, 2, \ldots$. By Lemma 2.7 $\{F_{n,k}\}_{n\ge 1}$ is equi-absolutely continuous for each k, where $F_{n,k}(x) = \int_a^x f_n \chi_{X_k}$ for each $x \in [a,b]$. By the Vitali convergence theorem $\int_a^b \left| f_n \chi_{X_k} - f_m \chi_{X_k} \right| \to 0$ as $n, m \to \infty$ for each k.

Set $k = 1$. There exists a subsequence $\{f_n^{(1)}\}$ of $\{f_n\}$ such that the series $h_1(x) = \sum_{n=1}^{\infty} \left| f_{n+1}^{(1)}(x)\chi_{X_1}(x) - f_n^{(1)}(x)\chi_{X_1}(x) \right|$ converges for almost all x in $[a,b]$ and $f_1^{(1)}(x)\chi_{X_1}(x) - h_1(x) \leq f_n^{(1)}(x)\chi_{X_1}(x) \leq f_1^{(1)}(x)\chi_{X_1}(x) + h_1(x)$ for almost all x in X_1 and for all n. Note that h_1 is Lebesgue integrable on $[a,b]$. Now consider $k = 2$, and the sequence $\{f_n^{(1)}\}$ in place of $\{f_n\}$. We obtain a subsequence $\{f_n^{(2)}\}$ of $\{f_n^{(1)}\}$ such that the series $h_2(x) = \sum_{n=1}^{\infty} \left| f_{n+1}^{(2)}(x)\chi_{X_2}(x) - f_n^{(2)}(x)\chi_{X_2} \right|$ converges for almost all x in $[a,b]$ and $f_1^{(2)}(x)\chi_{X_2}(x) - h_2(x) \leq f_n^{(2)}(x)\chi_{X_2}(x) \leq f_1^{(2)}(x)\chi_{X_2}(x) + h_2(x)$ for almost all x in X_2 and for all n. Continuing this process we get a subsequence $\{f_n^{(n)}\}$ of $\{f_n\}$ such that for each k, $g_k(x) \leq f_n^{(n)}(x) \leq h_k(x)$ for almost all x in X_k and g_k, h_k Lebesgue integrable on X_k; that is, condition (iii) of Theorem 3.4 holds for the sequence $\{f_n^{(n)}\}$. Since $\{F_n\} \in UACG_{ap}^*[a,b]$, by Lemma 3.1, $\{F_n\}$ satisfies $UASL$ on $[a,b]$. Hence condition (v) of Theorem 3.4 is satisfied. □

Theorem 3.6 *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ with primitives $\{F_n\}$ satisfying conditions (i), (ii), (iii), (iv) and (v) of Theorem 3.4. Then $\{F_n\} \in UACG_{ap}^*([a,b])$.*

PROOF. As in the proof of Theorem 3.3, $\{F_n\} \in UAC(X_k)$ for each k and so $\{F_n\} \in UACG(\bigcup_{k=1}^{\infty} X_k)$. By Lemma 3.2 for every $\epsilon > 0$ there exists a closed set $E \subseteq [a,b]$ with $|[a,b] \setminus E| < \epsilon$ such that $\{F_n\} \in UAC(E)$. By condition (v) of Theorem 3.4 $\{F_n\}$ satisfies $UASL$ on $[a,b]$. By Lemma 3.1 $\{F_n\} \in UACG_{ap}^*[a,b]$. □

We remark that Theorem 3.4 can't be proved by using the category argument as in the proof of Theorem 3.3. However it follows as a consequence of Theorem 3.6 and Lemma 3.1.

We shall give an example to show that Theorem 3.4 is indeed a genuine generalization of the dominated convergence theorem.

Example 3.7 Let

$$F(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}), & \text{when } 0 < x \leq 1. \\ 0, & \text{otherwise.} \end{cases}$$

Define $f_n(x) = F'(x)$ if $\frac{1}{n} \leq x \leq 1$ and $f_n(x) = 0$ otherwise. Then each f_n is AP integrable on $[0,1]$ with primitive F_n, say. Put $X_k = [\frac{1}{k+1}, 1]$, $k = 1, 2, \ldots$ and $N = \{0\}$. It is easy to verify that $\{f_n\}$ satisfy conditions (i), (ii), (iii), (iv) and (v) of Theorem 3.4. However f_n is not dominated by any AP integrable functions on the left or on the right.

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