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Dirk J. F. Nonnenmacher, Abteilung Mathematik II, Universität Ulm, D-89069 Ulm, Germany, (e-mail: nonnenma@mathematik.uni-ulm.de)

NEW INTEGRALS AND THE GAUSS–GREEN THEOREM WITH SINGULARITIES

The n-dimensional Gauss-Green theorem is a fundamental result in mathematical analysis and relates the integral $\int_A \operatorname{div}\vec{v}$ to the values of the vector field $\vec{v}$ on the boundary of A. Assuming $\vec{v}$ to be sufficiently smooth, its proof is well understood for smoothly bounded domains A, while its formulation in an ultimate generality requires some notion of exterior normal to a set A having no smoothness properties in a classical sense, cf. [1], [3]. It turns out that the *sets of finite perimeter* are the most general sets for which an exterior normal can profitably be defined and for which the divergence theorem holds for any continuously differentiable vector field on $\mathbb{R}^n$ having a compact support, cf. [4]. Thus from a geometrical point of view it is clear what kind of sets an optimal divergence theorem should handle.

In [2] the Gauss-Green theorem was established for Lipschitzian vector fields on $\mathbb{R}^n$ with compact support, and simple examples show that assuming $\vec{v}$ only to be differentiable on $\mathbb{R}^n$ then $\operatorname{div}\vec{v}$ might fail to be Lebesgue integrable on A. Thus, if we want to relax the assumption on the vector field further, we have either to impose some integrability assumption on the divergence or else to work with a more general integral.

Only recently several extensions of the Lebesgue integral were shown to integrate the divergence of any differentiable vector function over intervals, cf. [16], [8], [17], [21]. In [10] we developed an elementary self-contained axiomatic theory of non-absolutely convergent integral, and the resulting integral shows all the usual properties as linearity, additivity, extension of Lebesgue's integral etc. Specializing this concept leads to the well-behaved ν_1-integral over compact intervals [11], and in a corresponding divergence theorem we were able to treat *exceptional points*, i.e. points where $\vec{v}$ is not differentiable but still bounded, as well as *singularities* where $\vec{v}$ is only assumed to fulfill a local Lipschitz condition with a negative exponent $\beta > 1 - n$. In particular, this result contains all aforementioned ones.

In [18], using the ν_1-integral, we finally were able to extend this result to any bounded set of finite perimeter. In particular, no integrability assumptions are needed, and including exceptional points but no singularities similar results were obtained in [5], [6] [9], [19], [22]. Singularities on more general sets than intervals were treated only recently in our work [12], [14], cf. also [7].

Assuming in addition the divergence to be Lebesgue integrable on A our theorem will be free of the ν_1-integral and thus be part of the general Lebesgue theory. It is also noteworthy that our result can be obtained by using an integral which originally was only defined on intervals.

In order to keep our notation as simple as possible but still to give the reader a flavour of our results we here state a less general version of the theorem in [18], cf. [12], [15].

As usual we denote by $|\cdot|_s$ $(0 \le s \le n)$ the s-dimensional normalized outer Hausdorff measure in $\mathbb{R}^n$, and we call a set $E \subseteq \mathbb{R}^n$ to be σ_s-finite or to be an s-null set if it can be expressed as a countable union of sets with finite s-dimensional outer Hausdorff measure or if $|E|_s = 0$, respectively.

Suppose $A \subseteq \mathbb{R}^n$, $x \in A$, $1 - n \le \beta \le 1$ and let $\vec{v} : A \to \mathbb{R}^n$ be a vector function. We say that $\vec{v}$ satisfies at x the condition

(l_1) if there exists a real n by n matrix M such that

$$\vec{v}(y) - \vec{v}(x) - M(y - x) = o(1)||y - x|| \ (y \to x, \ y \in A) \ ,$$

(l_β) $(\beta \ne 1)$ if $\vec{v}(y) - \vec{v}(x) = o(1)||y - x||^\beta \ (y \to x, \ y \ne x, \ y \in A)$,

(L_β) if $\vec{v}(y) - \vec{v}(x) = O(1)||y - x||^\beta \ (y \to x, \ y \ne x, \ y \in A)$.

If $\vec{v}$ is differentiable at an interior point x of A, we set $\operatorname{div} \vec{v}(x) = \sum(\partial v_i / \partial x_i)(x)$, and at all other points $x \in A$ we set $\operatorname{div} \vec{v}(x) = 0$.

Gauss-Green Theorem. *Let A be a compact subset of $\mathbb{R}^n$ whose topological boundary ∂A has a finite $(n-1)$-dimensional measure and $\vec{v} : A \to \mathbb{R}^n$ be a vector function. By D we denote the set of all points from the interior of A where $\vec{v}$ is differentiable, and we express $A - D$ as a disjoint countable union of σ_{α_i}-finite sets M_i and α_i-null sets N_i with $0 < \alpha_i \le n$ $(i \in \mathbb{N})$ such that $\cup_{\alpha_i < n-1}(M_i \cup N_i)$ lies in the interior of A. If we assume that $\vec{v}$ satisfies the condition (l_{α_i+1-n}) and (L_{α_i+1-n}) at each point of M_i and N_i, respectively, then $\vec{v}$ is $|\cdot|_{n-1}$-measurable and bounded on ∂A, $\operatorname{div}\vec{v}$ is ν_1-integrable on A and*

$$\int_{\partial A} \vec{v} \cdot \vec{n}_A \, d|\cdot|_{n-1} = \overset{\nu_1}{\int_A} \operatorname{div} \vec{v} \, .$$

For a further discussion the reader should consult [18].

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