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ORDINARY AND STRONG DENSITY CONTINUOUS FUNCTIONS ON THE PLANE

This note is a summary of my lecture presented at the Eighteen Symposium on Real Analysis on June 1994. The proofs and more details on what follows can be found in [1] and [2]. The definitions of ordinary density and strong density topologies on the plane can be also found in [3] or [4].

The results presented below concern transformations from $\mathbb{R}^2$ into $\mathbb{R}$ and from $\mathbb{R}^2$ into $\mathbb{R}^2$ continuous with respect to different density topologies on the domain and range. We will consider *density (strong density) continuous* function $F\colon \mathbb{R}^2 \to \mathbb{R}$, i.e., continuous functions with the domain equipped with ordinary (strong) density topology and the range with one-dimensional density topology. The function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ studied below are *density (strong density) continuous*, i.e., continuous with respect to ordinary (strong) density topology on the domain and the range. We examine the relations between the (strong) density continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$ and of its sections $f(\cdot, y)$, $f(x, \cdot)$. Similar question is considered for transformations $F = (f,g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ and their coordinate functions f and g. We also discuss when the complex analytic functions are strongly and ordinarily density continuous.

Results concerning functions from $\mathbb{R}^2$ into $\mathbb{R}$:

Proposition 1 *[1] Let $h\colon \mathbb{R} \to \mathbb{R}$ and define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = h(y)$. The following conditions are equivalent.*

(a) *h is density continuous.*

(b) *f is density continuous.*

(c) *f is strongly density continuous.*

Proposition 2 *[1] If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is strongly density continuous then it is ordinarily density continuous.*

Example 1 *[1] There exists ordinary continuous $f\colon \mathbb{R}^2 \to \mathbb{R}$ with density continuous sections $f(x_0,\cdot)$ and $f(\cdot,y_0)$ for all $x_0, y_0 \in \mathbb{R}$ which is not ordinarily density continuous.*

Example 2 *[1] There exists continuous and ordinarily density continuous $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f(\cdot,0)$ is not density continuous.*

Theorem 1 *[1] If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is strongly density continuous then its sections $f(x_0,\cdot)$ and $f(\cdot,y_0)$ are density continuous for all $x_0, y_0 \in \mathbb{R}$.*

Results concerning transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$:

Proposition 3 *[1] Locally bi-Lipschitz transformations $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ are ordinarily density continuous.*

Theorem 2 *[1] Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $F(x,y) = (f(x,y), g(x,y))$.*

(a) *If F is strongly density continuous then f and g are also strongly density continuous.*

(b) *If F is density continuous then f and g are density continuous.*

Example 3 *[1] Transformation $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = (x, y^3)$, is not density continuous, while its coordinate functions $f(x) = x$ and $g(y) = y^3$ are density continuous.*

Example 4 *[1] Transformation $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = (x, x+y)$, is not strongly density continuous, while its coordinate functions $f(x,y) = x$ and $g(x,y) = x + y$ are strongly density continuous.*

Corollary 1 *[1] There exists a bi-Lipschitz transformations from $\mathbb{R}^2$ into $\mathbb{R}^2$ which is not strongly density continuous.*

Theorem 3 *[1] Let $f, g\colon \mathbb{R} \to \mathbb{R}$ and define transformation $H\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $H(x,y) = (f(x), g(y))$. If H is not constant then H is strongly density continuous if and only if functions f and g are density continuous and $\mathrm{m}_1\left(f^{-1}(p)\right) = \mathrm{m}_1\left(g^{-1}(p)\right) = 0$ for every $p \in \mathbb{R}$.*

Complex analytic functions:

Theorem 4 *[2] Every analytic function is ordinarily density continuous.*

Theorem 5 *[2] A non-constant analytic function F is strongly density continuous at z if and only if $F^{(n)}(z)$ is either real or imaginary number, where $n = \min\{k > 0\colon F^{(k)}(z) \neq 0\}$.*

Theorem 6 *[2] An analytic function F is strongly density continuous on its domain if and only if $F(z) = a + bz$ where $a, b \in \mathbb{R}^2 = \mathbf{C}$ and b is either real or imaginary number.*

References

[1] Krzysztof Ciesielski, Władysław Wilczyński, Density continuous transformations on $\mathbb{R}^2$, preprint.

[2] Krzysztof Ciesielski, Ordinary and strong density continuity of complex analytic functions, preprint.

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[4] Jaroslav Lukeš, Jan Malý, Luděk Zajíček, Fine Topology Methods in Real Analysis and Potential Theory, *Lecture Notes in Mathematics 1189*, Springer-Verlag 1986.