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PACKING CONICS IN THE PLANE

There exists a subset of the plane of Lebesgue measure zero, which contains a circle of every radius. This was first proved independently by Besicovitch & Rado, and by Kinney in 1968. Kinney gave direct constructions of such sets. Let C be a Cantor set. It is well known that $|C - C| = [0,1]$. For $r \in (0, \frac{1}{2}]$, let $(c_r, d_r) \in C\text{x}C$ denote the leftmost pair such that $r = \frac{d_r - c_r}{2}$. Then the set of circles centered at $(\frac{c_r + d_r}{2}, 0)$ and with radius r, obviously contains every radius between 0 and $\frac{1}{2}$. By taking countable union of similar sets one can get a set of circles of every radius. But to show that this set is actually of plane measure 0, Kinney is using theory of random walks. Still his proof is very complicated. (The paper is published in Amer. Math. Monthly!?) Four years later, Davies gives another "thin" set of circles containing every radius. Although his set is not as elementary as Kinney's set, his proof that the set is actually of plane measure zero is the simplest of these three, and in a nice way it is using the projection properties of irregular 1-sets.

In this paper we show using Davies' idea how one can easily construct a number of sets of plane measure zero containing

i) circles of every radius and centered at every point on the real line,

ii) circles of every radius (center at every point on the real line) and with the set of centers of Hausdorff dimension 0 (the set of radii of Hausdorff dimension 0),

iii) circles of every radius and passing through the origin,

iv) ellipses (hyperbolas) centered at every point on the real line and of every eccentricity,

v) ellipses (hyperbolas) centered at the origin and of every eccentricity.

Also in the same fashion we give a new proof that Kinney's set is of plane measure zero. Here we mention, that i), ii), iv) and v) do not follow directly from Davies' arguments.

Constructions of sets in i), ii), iii), iv) and v) are based on the following main Theorem

Theorem 1 *Let E be a compact irregular 1-set.*

a) Let $f : R^2 \to R$ and $g : R^2 \to R^+$ be C^∞ functions. Then, the set of circles centered at $(f(a,b),0)$, and with the radii $g(a,b)$ where $(a,b) \in E$ is of plane measure 0.

b) The set of circles centered at (a,b) and with the radii $\sqrt{a^2+b^2}$ where $(a,b) \in E$ is of plane measure zero.

c) The set of ellipses (hyperbolas) of the form $(u-p)^2 + k^2v^2 = p^2$ and of the form $u^2 + k^2v^2 = p^2$ ($(u-p)^2 - k^2v^2 = p^2$ and of the form $u^2 - k^2v^2 = p^2$), where $(p,k) \in E$ is of plane measure 0.

References

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