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ON HBV AND THE GARSIA–SAWYER CLASS

We consider continuous functions with domain $[a, b]$ and range $[c, d]$. Such a function f is said to be in the Garsia–Sawyer Class (GS) if $\int_c^d \log^+(n_f(y))\,dy$ is finite, where $n_f(y)$ is the Banach indicatrix of f. Garsia and Sawyer [2] showed that functions in GS have uniformly convergent Fourier series. Let $\Phi = \{\varphi_n\}$ be a sequence of convex functions with the following properties:

i) $\varphi_n : [0,\infty) \to [0,\infty)$ for $n = 1, 2, \ldots$;

ii) $\varphi_n(0) = 0$ and $\varphi_n(x) > 0$ for $x > 0,\ n = 1, 2, \ldots$;

iii) $\varphi_{n+1}(x) \le \varphi_n(x)$ for $x \ge 0,\ n = 1, 2, \ldots$;

iv) $\sum_{n=1}^{\infty} \varphi_n(x) = \infty$ for $x > 0$.

We have said [3] that f is of Φ-Bounded Variation (ΦBV) if there is a positive constant c so that $\sum \varphi_n\left(c\,|f(b_n) - f(a_n)|\right)$ is finite for any collection $\{[a_n, b_n]\}$ of non-overlapping subintervals of $[a, b]$ (and this is equivalent to requiring such sums to be uniformly bounded). By making appropriate choices of the functions φ_n, we may obtain many of the spaces of generalized bounded variation that have been studied. In particular, if $\varphi_n(x) = x/n$, we have the functions of Harmonic Bounded Variation (HBV), introduced by Waterman [4] (in this case we may take $c = 1$ above). Waterman showed that continuous functions in HBV have uniformly convergent Fourier series, and moreover [5] that GS $\subseteq$ HBV. HBV is pivotal in this context, since if ΦBV properly contains HBV, there is a continuous function in ΦBV whose Fourier series diverges at a point. But GS is not closed under addition, so GS is not the same as HBV (an illustration of this fact may be found in [1]). The full story of the relationship between GS and HBV is not yet known. Here we establish a result relating to the way GS is distributed through HBV.

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Theorem 1 *If ΦBV is properly contained in HBV, there are*

i) a function in GS that is not in ΦBV;

ii) a function in $HBV \setminus \Phi BV$ that is not in GS.

PROOF. i) If ΦBV is properly contained in HBV, there is a sequence $\{\alpha_n\}$ so that $\sum_{n=1}^{\infty} \varphi_n(c\alpha_n) = \infty$ for any $c > 0$ while $\sum_{n=1}^{\infty} \alpha_n/n < \infty$. We may assume that $\alpha_{2n-1} = \alpha_{2n}$ and $\alpha_{2n+1} < \alpha_{2n}, n = 1, 2, \ldots$. Let $\beta_n = \alpha_{2n-1} = \alpha_{2n}, n = 1, 2, \ldots$. Let f be the piecewise linear function that passes through the points $(0,0), (1,0), (1/2^n, 0)$, and $(3/2^{n+1}, \beta_n)$, for $n = 1, 2, \ldots$, and has no corners other than at these points. Then $f \in$ HBV on [0, 1], as $\sum \alpha_n/n < \infty$, and we obtain the supremum of the sums $\sum |f(b_n) - f(a_n)|/n$ by choosing intervals $\{[a_n, b_n]\}$ in such a way that $|f(b_n) - f(a_n)| = \alpha_n$. But $f \notin \Phi$BV, since the same choice of $\{[a_n, b_n]\}$ yields, for any $c > 0$, the sum $\sum \varphi_n(c\alpha_n) = \infty$.

We now show that $f \in$ GS. The range of f is $[0, \beta_1]$, and $n_f(y)$ is a step function whose value changes only when $y = \beta_2, \beta_3, \ldots$. If $\beta_{n+1} < y < \beta_n$, we have $n_f(y) = 2n$ (we need not be concerned with the values of n_f on the (countable) set 0, $\beta_1, \beta_2, \ldots$). Thus

$$\int_0^{\beta_1} \log^+(n_f(y))\, dy = \lim_{k\to\infty} \int_{\beta_k}^{\beta_1} \log^+(n_f(y))\, dy = \lim_{k\to\infty} \sum_{n=1}^{k} \log(2n)(\beta_n - \beta_{n+1}).$$

Applying summation by parts to the last expression, we obtain:

$$\sum_{n=1}^{k} \log(2n)(\beta_n - \beta_{n+1})$$
$$= \beta_1 \log(2) - \beta_{k+1} \log(2(k+1)) + \sum_{n=2}^{k+1} \beta_n(\log(2n) - \log(2(n-1)))$$

Now $\beta_k \log(k) < \beta_k \sum_{n=1}^{k} 1/n \leq \sum_{n=1}^{k} \beta_n/n$, which sums are uniformly bounded since $f \in$ HBV (these sums are obtained with the proper choice of intervals in the definition of HBV). Thus $\beta_{k+1} \log(2(k+1))$ is bounded. By the Mean Value Theorem, we have

$$\sum_{n=2}^{k+1} \beta_n(\log(2n) - \log(2(n-1))) = \sum_{n=2}^{k+1} \beta_n/\xi_n$$

where $2(n-1) < \xi_n < 2n$. But then $\sum_{n=2}^{k+1} \beta_n/\xi_n < \sum_{n=2}^{k+1} \beta_n/(2(n-1))$, which sums are also uniformly bounded. It follows that $f \in$ GS.

ii) Let g be any function in HBV\GS (for instance, the one constructed in [1]). Clearly, g cannot be monotone, so there exist $a^* \neq b^* \in [a,b]$ with $g(a^*) = g(b^*)$. We construct a function g^* as follows. Let $g^*(x) = g(x)$ for $x \in [a,a^*] \cup [b^*,b]$. On the interval $[a^*,(a^*+b^*)/2]$, the values of g^* will be those of g on all of $[a^*,b^*]$, and on $[(a^*+b^*)/2,b^*]$, we let $g^*(x) = g(a^*)$. Then $g^* \in$ HBV, and since $n_g(y) = n_{g^*}(y)$ for all y except possibly $y = g(a^*)$, we have $g^* \notin$ GS. Furthermore, as long as we maintain continuity, we may alter the values of g^* on $[(a^*+b^*)/2,b^*]$ in any manner at all without negating the latter fact.

Now we will do just that. With f as in the first half of the proof, let

$$f^*(x) = \begin{cases} f\left(\frac{2}{b^*-a^*}\left(x - \frac{a^*+b^*}{2}\right)\right) & \frac{a^*+b^*}{2} < x < b^* \\ 0 & \text{otherwise} \end{cases}$$

Then $f^* \in$ HBV\ΦBV. Since $f^*, g^* \in$ HBV, we have $f^* + g^* \in$ HBV. Since $f \notin \Phi$BV, we have $f^* + g^* \notin \Phi$BV (the sums that demonstrate that $f \notin \Phi$BV are obtainable from f^*+g^* with the appropriate choice of intervals). Finally, it follows from the comment at the end of the previous paragraph that $f^* + g^* \notin$ GS. □

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