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A POTENTIAL THEORETIC PROOF OF AN INEQUALITY OF C. D. CUTLER AND L. OLSEN

Abstract

We present a potential theoretic proof of the following inequality of C. D. Cutler and L. Olsen [C. D. Cutler & L. Olsen, *A Variational Principle for the Hausdorff Dimension of Fractal Sets*, Math. Scand. (to appear)]: If E is a Borel subset of $\mathbb{R}^d$, then

$$\dim E \leq \sup_{\mu \in \mathcal{P}(E)} \underline{R}(\mu)$$

where $\dim E$ denotes the Hausdorff dimension of E, $\mathcal{P}(E)$ denotes the family of Borel probability measures supported by E, and $\underline{R}(\mu)$ denotes the lower Rényi dimension of the measure μ.

1. Definitions and Statement of Result.

Let X be a separable metric space, $E \subseteq X$ and $s \geq 0$. Then the s-dimensional Hausdorff measure $\mathcal{H}^s(E)$ of E is defined by

$$\mathcal{H}^s(E) = \sup_{\delta>0} \inf\{\sum_{i=1}^{\infty} (\text{diam } E_i)^s \mid E \subseteq \cup_{i=1}^{\infty} E_i \text{ , diam } E_i < \delta \quad \text{for all } i \in \mathbb{N}\}.$$

The Hausdorff dimension $\dim E$ of E is defined by

$$\dim E = \inf\{s \geq 0 \mid \mathcal{H}^s(E) < \infty\} = \sup\{s \geq 0 \mid \mathcal{H}^s(E) > 0\}.$$

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The s-dimensional packing measure $\mathcal{P}^s(E)$ of E is defined in two stages. For $x \in X$ and $r > 0$, $D(x,r)$ denotes the closed ball in X with center x and radius r. Now put

$$\begin{aligned}\overline{\mathcal{P}}^s(E) &= \inf_{\delta>0} \sup\{\sum_{i=1}^{\infty}(\text{diam } D(x_i,r_i))^s \mid (D(x_i,r_i))_{i\in\mathbb{N}} \\ &\qquad \text{is a pairwise disjoint family, } x_i \in E \text{ , } r_i < \delta\}\,.\end{aligned}$$

Then $\mathcal{P}^s(E) = \inf_{E\subseteq\cup_{i=1}^{\infty}E_i} \sum_{i=1}^{\infty} \overline{\mathcal{P}}^s(E_i)$. The packing dimension Dim E of E is defined by

$$\text{Dim } E = \inf\{s \geq 0 \mid \mathcal{P}^s(E) < \infty\} = \sup\{s \geq 0 \mid \mathcal{P}^s(E) > 0\}\,.$$

It is well-known that $\dim E \leq \text{Dim } E$ for all $E \subseteq \mathbb{R}^d$.

We will now define the Rényi dimension. Let $\mathcal{P}(X)$ denote the family of Borel probability measures on X. Fix $\mu \in \mathcal{P}(X)$ and write

$$\begin{aligned}h_r(\mu) &= \inf\{-\sum_{i=1}^{\infty}\mu(E_i)\log\mu(E_i) \mid (E_i)_{i\in\mathbb{N}} \\ &\qquad \text{is a Borel partition of } X, \text{ diam } E_i < r\}\end{aligned}$$

for $r > 0$. The upper and lower Rényi dimensions of μ are then defined by $\overline{R}(\mu) = \limsup_{r\searrow 0} -\frac{h_r(\mu)}{\log r}$ and $\underline{R}(\mu) = \liminf_{r\searrow 0} -\frac{h_r(\mu)}{\log r}$ respectively, cf.[0].

Cutler & Olsen [0] proved the following two inequalities.

Theorem 1 ([0, Proposition 5]) *Let $E \subseteq \mathbb{R}^d$. Then the following assertions hold:*

i) If E is a Borel set, then $\dim E \leq \sup_{\mu\in\mathcal{P}(E)} \underline{R}(\mu)$.

ii) If E is a bounded Borel set, then $\sup_{\mu\in\mathcal{P}(E)} \overline{R}(\mu) \leq$ *Dim E.*

The proof of Theorem 1, part i), in [0] is based on a characterization (due to Tricot [0, Theorem 1, p. 62]) of $\dim E$ in terms of "local dimensions" of measures supported by E. In this note we present a potential theoretic proof of the inequality in Theorem 1, part i).

2. Proof.

For $x \in \mathbb{R}^d$ and $r > 0$, $B(x,r)$ denotes the Euclidean ball in $\mathbb{R}^d$ with center x and radius r. Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $t \geq 0$. We define the t-potential $\Phi_t(\mu;x)$

of μ at a point $x \in \mathbb{R}^d$ by $\Phi_t(\mu;x) = \int \frac{1}{\|x-y\|^t}\,d\mu(y)$ where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Next we define the t-energy $I_t(\mu)$ of μ by $I_t(\mu) = \int \Phi_t(\mu;x)\,d\mu(x)$. The reader is referred to [0] for a discussion of the t-potential and the t-energy.

We will need the following two results in order to prove Theorem 1, part i).

Theorem 2 (Frostman) *Let E be a Souslin subset of $\mathbb{R}^d$. If $0 \le t < \dim E$, then there exists a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ with support in E such that $I_t(\mu) < \infty$.*

Proof. See [0, Corollary 6.6]. □

Lemma 1 *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$, E be a Borel subset of $\mathbb{R}^d$ and $A, t \ge 0$. If*

$$\log \mu(B(x,r)) \le A + t\log r$$

for all $x \in E$ and $r > 0$, then $\underline{R}(\mu) \ge \mu(E)t$

Remark. Lemma 1 represents a slight generalization of [0, Lemma 2], and the proof is similar to the proof of [0, Lemma 2]. However, we include the proof here for sake of completeness.

Proof. Choose $r_0 > 0$ such that $t\log r + A < 0$ for $0 < r < r_0$. Let $0 < r < \min(r_0, 1)$ and $(E_i)_i$ be a partition of $\mathbb{R}^d$ with diam $E_i < r$ for all i. Write $I = \{i \mid E_i \cap E \neq \emptyset\}$. For $i \in I$ choose a point $x_i \in E_i \cap E$ and observe that $E_i \subseteq B(x_i, r)$. Hence

$$\log \mu(E_i) \le \log \mu(B(x_i,r)) \le t\log r + A \quad \text{for } i \in I. \tag{1}$$

It follows from (1) that

$$\begin{aligned} -\sum_i \mu(E_i)\log\mu(E_i) &\ge -\sum_{i\in I}\mu(E_i)\log\mu(E_i) \ge -\sum_{i\in I}\mu(E_i)(t\log r + A) \\ &= -\mu(\cup_{i\in I}E_i)(t\log r + A) \ge -\mu(E)(t\log r + A) \end{aligned}$$

Since the partition $(E_i)_i$ was arbitrary this inequality implies that

$$h_r(\mu) \ge -\mu(E)(t\log r + A) \quad \text{for } 0 < r < \min(r_0, 1).$$

Hence $\underline{R}(\mu) = \liminf_{r\searrow 0} -\frac{h_r(\mu)}{\log r} \ge t\mu(E)$. □

We are now ready to give a potential theoretic proof of Theorem 1, part i).

Potential theoretic proof of Theorem 1, part i). We may clearly assume that $\dim E > 0$. Now fix $0 \le t < \dim E$ and $\varepsilon > 0$. It follows from Theorem 2 that there exists a measure $\mu \in \mathcal{P}(E)$ satisfying $\int \Phi_t(\mu; x)\, d\mu(x) = I_t(\mu) < \infty$. Hence

$$\int \frac{1}{\|x-y\|^t}\, d\mu(y) = \Phi_t(\mu; x) < \infty \quad \text{for } \mu - \text{a.a. } x \in \mathbb{R}^d . \tag{2}$$

For $n \in \mathbb{N}$ write $E_n = \{x \in \mathbb{R}^d \mid \int \frac{1}{\|x-y\|^t}\, d\mu(y) \le n\}$. Clearly $E_n \nearrow \cup_m E_m$, and $\mu(\cup_m E_m) = 1$ by (2). We can thus choose an integer $N \in \mathbb{N}$ satisfying $\mu(E_N) \ge 1 - \varepsilon$. Next observe that all $x \in E_N$ and $r > 0$ satisfy,

$$\begin{aligned} N &\ge \int \frac{1}{\|x-y\|^t}\, d\mu(y) \ge \int_{B(x,r)} \frac{1}{\|x-y\|^t}\, d\mu(y) \\ &\ge \int_{B(x,r)} \frac{1}{r^t}\, d\mu(y) = \mu(B(x,r)) r^{-t} . \end{aligned}$$

Hence $t \log r + \log N \ge \log \mu(B(x,r))$ for $x \in E_N$ and $r > 0$. An application of Lemma 1 now yields

$$\sup_{\nu \in \mathcal{P}(E)} \underline{R}(\nu) \ge \underline{R}(\mu) \ge \mu(E_N) t \ge (1-\varepsilon) t$$

which completes the proof since $t < \dim E$ and $\varepsilon > 0$ were arbitrary. □

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