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Roy A. Mimna, 520 Meadowland Drive, Hubbard, OH 44425

A NOTE ON FUNCTIONS DETERMINED BY DENSE SETS

Abstract

This paper deals with the question of finding sets of functions which are determined by their values on dense sets. The main result gives a new condition sufficient for a class of functions to have that property. The condition implies the known facts that approximately differentiable functions and separately continuous functions are determined by dense sets. Also presented is the rèsult that approximately differentiable functions on the unit interval are quasi-continuous. A lemma is given showing that if X is a topological space, and D is a dense semi-closed subset of X, then D is the whole space X.

1. INTRODUCTION

It is well known that continuous functions on a space X into a Hausdorff space Y, are determined by their values on dense sets. C. Neugebauer has shown that approximately differentiable real-valued functions on the unit interval, are determined by their values on dense sets. Here we present a larger class of functions which are determined by their values on dense sets, a class which includes approximately differentiable functions and separately continuous functions.

2. APPROXIMATELY DIFFERENTIABLE FUNCTIONS AND QUASI-CONTINUITY

Definition 1 *Let $f : [0,1] \to \mathbb{R}$ be a real-valued function, where the interval $[0,1]$ has the usual subspace topology. Let x' be a point in $[0,1]$. If there*

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exists a set E such that (1) x' is in E; (2) x' is a point of zero density with respect to the complement of E; and (3) the limit $\lim_{x\to x'} \frac{f(x)-f(x')}{x-x'}$ exists for x restricted to E, then this limit is the approximate derivative of f at x' and is written $f'_{ap}(x')$. If $x'=0$ or $x'=1$, the obvious modifications can be made using limits from the right and from the left.

Real-valued functions which are approximately differentiable on $[0,1]$ are determined by their values on dense sets. [9]

Following Goffman and Waterman [8], we say that a function $f:[0,1]\to\mathbb{R}$ is *approximately continuous* at x' if for any open set V containing $f(x')$, the set $f^{-1}(V)$ has metric density one at x'. That is, for any sequence $\{I_n\}_{n=1}^{\infty}$ of intervals such that $\lim_{n\to\infty}|I_n| = 0$ and x' is in I_n for all n, $\lim_{n\to\infty}\frac{|f^{-1}(V)\bigcap I_n|}{|I_n|} = 1$.

Goffman and Neugebauer have shown that if $f:[0,1]\to\mathbb{R}$ has an approximate derivative everywhere on $[0,1]$, and if $E = \{x \text{ in } [0,1] : f'(x) \text{ exists}\}$, then for every subinterval I of $[0,1]$, $I\bigcap E$ contains an interval. [6] We apply this result in the proof of Theorem 1 below. First, we define the notion of quasi-continuity:

Definition 2 *Let $f: X\to Y$ be a function on a topological space X into a topological space Y. Let x be a point in X. We say that f is quasi-continuous at x iff for any open set V containing $f(x)$ and for any open set U containing x, there exists a nonempty open set G, contained in U, such that $f(x)$ is in V for all x in G. If f is quasi-continuous at x for all x in X, we say that f is quasi-continuous.*

Theorem 1 *Let $f:[0,1]\to\mathbb{R}$ be a real-valued function. If f is approximately differentiable on $[0,1]$, then f is quasi-continuous.*

PROOF. Let x' be a point in $[0,1]$. Let V be an open set containing $f(x')$ and let U be an open set containing x'. There exists an interval I in U such that $x'\in I\subset U$. Since f is approximately continuous at x', the set $f^{-1}(V)$ has density one at x'. There is a sequence $\{I_n\}_{n=1}^{\infty}$ of intervals containing x' such that $\lim_{n\to\infty}|I_n| = 0$. Then for some n_0 in $\mathbb{N}$, $I_n\subset I$ for all $n\geq n_0$. By the theorem of Goffman and Neugebauer, each I_n in $\{I_n\}_{n=1}^{\infty}$ contains an interval J_n such that $J_n\subset E=\{x : f'(x) \text{ exists}\}$. The claim is that $f^{-1}(V)\bigcap J_n\neq\phi$ for some $n, n\geq n_0$. Assume that $f^{-1}(V)\bigcap J_n=\phi$ for all $n, n\geq n_0$. Then $|f^{-1}(V)\bigcap J_n| = 0$ for all $n, n\geq n_0$. Since J_n is contained in I_n for every n, and since $|f^{-1}(V)\bigcap J_n| = 0$, $|f^{-1}(V)\bigcap I_n| = |(f^{-1}(V)\bigcap(I_n\backslash J_n))\bigcup(f^{-1}(V)\bigcap J_n)| = |f^{-1}(V)\bigcap(I_n\backslash J_n)|$. It follows that $\lim_{n\to\infty}\frac{|f^{-1}(V)\bigcap(I_n\backslash J_n)|}{|I_n|} = 1$. But this is impossible. Therefore, for some

$n, n \geq n_0$, J_n contains a point of $f^{-1}(V)$, and since f is continuous on J_n, there is an interval J_0 in J_n such that $f(x)$ is in V for all x in J_0. Hence f is quasi-continuous. $\square$

A well known property of quasi-continuous functions is that inverse images of open sets are semi-open and inverse images of closed sets are semi-closed. [10]

Definition 3 *A set A in a topological space X is semi-open iff there exists an open set G such that $G \subset A \subset \overline{G}$, there $(^-)$ denotes the closure operator. A set B in X is semi-closed iff its complement is semi-open.*

Crossley and Hildebrand defined the *semi-closure* of a set S in a topological space X, denoted by $\underline{S}$, to be the intersection of all semi-closed sets containing S. Then the set S is semi-closed iff $S = \underline{S}$. [3]

Theorem 2 *Let f and g be approximately differentiable functions of the form $f, g : [0,1] \to \mathbb{R}$. Then the set $A = \{x \text{ in } [0,1] : f(x) = g(x)\}$ is semi-closed in $[0,1]$.*

PROOF. Since the set of approximately differentiable functions is closed under subtraction (see [9]), we may define the function $h : [0,1] \to \mathbb{R}$ such that $h(x) = f(x) - g(x)$ for all x in $[0,1]$. Then $h(x) = 0$ for all x in A. By Theorem 1, h is quasi-continuous; thus, $h^{-1}(\{0\})$ is semi-closed in $[0,1]$. Also, $A \subset h^{-1}(\{0\})$. Now $\underline{A}$ is the smallest semi-closed set containing A. It follows that $\underline{A} \subset h^{-1}(\{0\})$ and $h(\underline{A}) \subset h[h^{-1}(\{0\})] = \{0\}$. Hence, $A = \underline{A}$ and A is semi-closed. $\square$

Lemma 1 *Let X be a topological space. Let D be a dense subset of X. If D is semi-closed in X, then $D = X$.*

PROOF. Since D is semi-closed, the complement $(X \backslash D)$ is semi-open. Then there exists an open set G in X such that $G \subset (X \backslash D) \subset \overline{G}$. Since G is open and D is dense in X, then if G is not empty, $G \bigcap D$ is not empty. But this is impossible. So, G must be the empty set. Thus, $\phi \subset (X \backslash D) \subset \overline{\phi} = \phi$ and $D = X$. $\square$

3. CLASSES OF FUNCTIONS DETERMINED ON DENSE SETS

From the lemma we immediately have the following:

Theorem 3 *Let S be the set of real-valued functions of the form $f : X \to \mathbb{R}$, where X is a topological space. Suppose that for any functions f and g in S, the set $A = \{x \text{ in } X : f(x) = g(x)\}$ is semi-closed in X. Then the functions in S are determined by their values on dense sets.*

Corollary 1 *Let f and g be functions of the form $f, g : [0,1] \to \mathbb{R}$ which are approximately differentiable on $[0,1]$. If f and g agree on a dense subset D of $[0,1]$, then $f \equiv g$.*

The above result was proved by C. Neugebauer in [9].

Corollary 2 *Let f and g be functions of the form $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which are separately continuous (i.e., continuous in each variable separately). If f and g agree on a dense subset D of $\mathbb{R} \times \mathbb{R}$, then $f \equiv g$.*

PROOF. Since separately continuous real-valued functions are closed under subtraction, and are quasi-continuous, we can mimic the proof of Theorem 2 to show that the set $A = \{(x,y) : f((x,y)) = g((x,y))\}$ is semi-closed. □

For a much more general result on separate continuity, see [7].

Remark 1 *Quasi-continuous functions are not determined by dense sets. [5] Also, bilaterally quasi-continuous functions (see [4]) are not determined on dense sets. For example, consider the function $f : [-1,1] \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Remark 2 *Separately continuous functions are not approximately continuous. For example, consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by*

$$f((x,y)) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}.$$

Let V be the open interval $(-\frac{1}{4}, \frac{1}{4})$. Then the point $(0,0)$ is not a point of density one with respect to the set $f^{-1}(V)$. Approximately continuous functions are not determined on dense sets. [9]

Remark 3 *In addition to approximately differentiable functions and separately continuous functions, the class of functions described in Theorem 3 contains other sets of functions. For example, let S be the set of functions of the form $f : [0,1] \to \mathbb{R}$, where for each f in S, there is a finite subdivision, $0 = a_1 < a_2 < \cdots < a_n = 1$, of the interval $[0,1]$, and there are real numbers $k_i, 1 \leq i \leq n-1$, such that*

$$f(x) = \begin{cases} k_i & \text{for } a_i \leq x < a_{i+1} \\ k_{n-1} & \text{for } x = 1 \end{cases}.$$

Since the functions in S are quasi-continuous <u>*and*</u> *closed under subtraction, by Theorem 3, they are determined by dense sets. However, the functions in S are not approximately differentiable. The reader will readily find similar examples of sets of functions of the form $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where the functions are determined by dense sets but are not separately continuous.*

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