Real Analysis Exchange
Vol. 19(2), 1993/94, pp. 644–650

D. Rinne*, Department of Mathematics, California State Univ., San Bernardino, CA, 92407

RECTANGULAR AND ITERATED CONVERGENCE OF MULTIPLE TRIGONOMETRIC SERIES

Abstract

In this paper we present a proof of SH. T. Tetunashvili that shows that a multiple trigonometric series that converges rectangularly everywhere actually converges iteratively everywhere to the same function. This method then solves a uniqueness problem, namely, that if a multiple trigonometric series converges rectangularly everywhere to zero, then all the coefficients are zero. We give a detailed proof in two dimensions. The result for higher dimensions may then be obtained inductively using the same proof.

The purpose of this article is to explain how a technique used by SH. T. Tetunashvili [5] gives a short easy proof of a uniqueness problem for multiple trigonometric series, namely, that if a multiple trigonometric series converges rectangularly everywhere to zero, then all the coefficients are zero. We give a detailed proof in two dimensions. This case contains the essential ideas and it will then be easy for the reader to see how to induct to higher dimensions. Although Tetunashvili worked in a more general setting, by reducing the problem to that of a multiple trig series that converges rectangularly everywhere to a finite function, we get the nice result that the series actually has iterated convergence everywhere to the same function, thus making uniqueness of the coefficients an easy matter by one-dimensional uniqueness. We also make the observation that iterated convergence everywhere is equivalent to a weakened form of rectangular convergence everywhere.

*This author partially supported by NSF.

Key Words: Unrestricted rectangular convergence, iterated convergence, uniqueness

Mathematical Reviews subject classification: Primary: 42B05, 42A63

Received by the editors December 1, 1993

1. Definitions and Notation

Let Z be the integers and Z_0 the non-negative integers. We consider multiple trig series in d dimensions (d≥2) of the form

$$(1) \qquad T(X) = \sum a_M e^{iMX}$$

where $M = (m_1, ..., m_d) \in Z^d$, $X = (x_1, ..., x_d) \in [0,2\pi]^{11d}$, MX means $M \cdot X$ and the sum is over some collection of values of M. For $K = (k_1, ..., k_d) \in Z_0^d$ we use $\|K\|$ to denote $\min_i\{k_i\}$. By rectangular convergence we mean what is usually called unrestricted rectangular convergence and is defined as follows. The rectangular partial sums are of the form

$$(2) \qquad S_K(X) = \sum_{m_1=-k_1}^{k_1} \cdots\cdots \sum_{m_d=-k_d}^{k_d} a_M e^{iMX}$$

and the series in (1) converges unrestricted rectangularly to the function $f(X)$ if $\lim_{\|K\|\to\infty} S_K(X) = f(X)$. Related notions of convergence can be obtained as follows. If, in (2), we require that $\frac{\max_i\{k_i\}}{\min_i\{k_i\}}$ be bounded and we get the same limit for every bound, then we have restricted rectangular convergence. By making all k_i equal we get square convergence. Obviously, rectangular convergence implies restricted rectangular convergence which implies square convergence. If

$$(3) \qquad \sum_{m_1=-\infty}^{\infty} \cdots\cdots \sum_{m_d=-\infty}^{\infty} a_M e^{iMX}$$

converges when the d infinite sums are iterated from right to left (where $\sum_{m_i=-\infty}^{\infty}$ means $\lim_{k_i\to\infty} \sum_{m_i=-k_i}^{k_i}$), then we say that the sum converges iteratively. Numerous examples that compare these and other methods of convergence can be found in Ash-Welland [2].

2. History

The proof of uniqueness in one dimension is due to Cantor [3] in 1870. In 1972, Ash and Welland [2] proved uniqueness in dimension two. However, their proof used results of V. Shapiro on spherical Abel summability in dimension two and did not generalize to higher dimensions. In 1993, Ash, Freiling and

Rinne [1] answered the uniqueness question in higher dimensions. The general approach of their paper was to show that if a trig series converges rectangularly everywhere to 0, we can show that the second formal integral of that series has a certain generalized derivative that is also 0, and that this is strong enough to force all the coefficients to be zero. The work of Tetunashvili, which appeared in English translation in 1992, avoids the complicated calculations in [1] by inductively converting the problem to an application of the known one-dimensional uniqueness. It is this argument which we will explain in the next section.

3. Main Results

We first include a proof of uniqueness for iterated convergence since it is short and shows clearly how the one-dimensional uniqueness enters the picture.

Theorem 1 *If* $\sum_{m_1=-\infty}^{\infty} \cdots\cdots \sum_{m_d=-\infty}^{\infty} a_M e^{iMX}$ *converges iteratively everywhere to zero, then all coefficients are zero.*

PROOF. The case d=1 is Cantor's Theorem. Suppose the theorem holds in dimension d-1 and $\sum_{m_1=-\infty}^{\infty} \cdots\cdots \sum_{m_d=-\infty}^{\infty} a_M e^{iMX}$ converges everywhere to zero. For x_d fixed, we have

$$\sum_{m_1=-\infty}^{\infty} \cdots\cdots \sum_{m_{d-1}=-\infty}^{\infty} \sum_{m_d=-\infty}^{\infty} a_M e^{iMX} = \sum_{m_1=-\infty}^{\infty} \cdots\cdots \sum_{m_{d-1}=-\infty}^{\infty} b_{M'} e^{iM'X'}$$

where $M' = (m_1, ..., m_{d-1})$, $X' = (x_1, ..., x_{d-1})$ and $b_{M'} = \sum_{m_d=-\infty}^{\infty} a_M e^{im_d x_d}$. Then $\sum_{m_1=-\infty}^{\infty} \cdots\cdots \sum_{m_{d-1}=-\infty}^{\infty} b_{M'} e^{iM'X'}$ converges everywhere to zero, so all $b_{M'}$ are zero by hypothesis. Since x_d was arbitrary $\sum_{m_d=-\infty}^{\infty} a_M e^{im_d x_d}$ is zero for all x_d. By Cantor's Theorem, all a_M are zero.□

Starting with techniques developed by P. Cohen in his thesis [4] to estimate the rate of increase of the coefficients of a multiple trigonometric series under certain types of convergence, Ash and Welland [2] proved the following lemma.

Lemma 2 *If the multiple trig series* $T(X) = \sum a_M e^{iMX}$ *converges rectangularly everywhere, then, for each* x_d *and for each* $(m_1, ..., m_{d-1})$, *all sums of the form* $\sum_{m_d=-k_d}^{k_d} a_M e^{im_d x_d}$ *are bounded.*

The next theorem, although not explicitly stated in [5], is a direct consequence of the work there and the proof we give is essentially that of Tetunashvili, made simpler by the assumption of convergence everywhere.

Theorem 3 *Suppose a 2-dimensional trig series* $\sum a_{mn}e^{i(ny+mx)}$ *converges rectangularly everywhere to the finite function* $f(x,y)$. *Then the corresponding iterated series* $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn}e^{i(ny+mx)}$ *converges everywhere to* $f(x,y)$.

PROOF. Let $T(x,y) = \sum a_{mn}e^{i(ny+mx)}$ be a 2-dimensional trig series which converges rectangularly everywhere to the function $f(x,y)$. The rectangular sums will be written $S_{kj}(x,y) = \sum_{m=-k}^{k} \sum_{n=-j}^{j} a_{mn}e^{i(ny+mx)}$. Fix y and for each m consider sums of the form $\sum_{n=-j}^{j} a_{mn}e^{iny}$, bounded over all j by Lemma 2. For this same fixed y, let $\{j_i^0\}$ be a subsequence of the natural numbers so that $\lim_{i\to\infty} \sum_{n=-j_i^0}^{j_i^0} a_{0n}e^{iny}$ exists and call this limit b_0. We proceed inductively. Suppose b_l, b_{-l} and $\{j_i^l\} = \{j_i^{-l}\}$ have been chosen for $0 \le l \le m-1$. Let $\{j_i^m\} = \{j_i^{-m}\}$ be a subsequence of $\{j_i^{m-1}\}$ so that $\lim_{i\to\infty} \sum_{n=-j_i^m}^{j_i^m} a_{mn}e^{iny}$ and $\lim_{i\to\infty} \sum_{n=-j_i^{-m}}^{j_i^{-m}} a_{-mn}e^{iny}$ exist and call these limits b_m and b_{-m} respectively. We now show that for each y, $T(x) = \sum_{m=-\infty}^{\infty} b_m e^{imx}$ converges to $f(x,y)$. Fix x and let $\epsilon > 0$. Pick K large enough so that $\|(k,j)\| \ge K$ implies $S_{kj}(x,y)$ is within $\epsilon/2$ of $f(x,y)$. Let $k \ge K$ and pick j to be a sufficiently large term of the sequence $\{j_i^k\}$ so that $\left| b_m - \sum_{n=-j}^{j} a_{mn}e^{iny} \right| < \epsilon/2^{|m|+3}$ for $-k \le m \le k$. Then

$$\left| \sum_{m=-k}^{k} b_m e^{imx} - f(x,y) \right| \le$$

$$\left| \sum_{m=-k}^{k} (b_m - \sum_{n=-j}^{j} a_{mn}e^{iny}) e^{imx} \right| + |S_{kj}(x,y) - f(x,y)| \le$$

$$\sum_{m=-k}^{k} \epsilon/2^{|m|+3} + \epsilon/2 < \epsilon,$$

as desired.

Since we now have convergence of $T(x)$, the coefficients b_m are unique. This of course says that only one choice existed for b_0. Thus, we get the convergent series $\sum_{n=-\infty}^{\infty} a_{0n}e^{iny} = b_0$. Observe, however, that by changing the starting m value, from 0 to any integer we choose, in the inductive construction of the b_m, $\sum_{n=-\infty}^{\infty} a_{mn}e^{iny}$ converges for every m and we may replace b_m with $\sum_{n=-\infty}^{\infty} a_{mn}e^{iny}$ to get iterated convergence of the original series to $f(x,y)$ for this y. Since y was arbitrary, the theorem is proved. □

Uniqueness of coefficients now follows since rectangular convergence everywhere to zero becomes iterated convergence everywhere to zero.

We should state the corresponding theorem from Tetunashvili [5] as it is given there to give the reader an idea of the more general setting used. We use $\Phi = \{\varphi_i\}_{i=0}^{\infty}$ to denote a general sequence of bounded real functions defined on [0,1], and use $T^1 = \{t_i\}_{i=0}^{\infty}$ to represent the trigonometric system on [0,1] ordered to start with constant 1 and then alternate cosines and sines of decreasing periods. T^d is then the d-fold trig system. A measurable subset A of [0,1] is in class $U(\Phi)$ if convergence of $\sum_{i=0}^{\infty} a_i\varphi_i(x)$ to zero on A implies all coefficients are zero. Then $U(T^{d-1} \times \Phi)$ is defined inductively by $E \in U(T^{d-1} \times \Phi)$ if $E = \{(x_1, ..., x_d) : x_d \in A \text{ and } E_{(x_d)} \in U(T^{d-1})\}$ where $A \in U(\Phi)$ and $E_{(x_d)} = \{(x_1, ..., x_{d-1}) : (x_1, ..., x_{d-1}, x_d) \in E\}$. Let μ_d be d-dimensional Lebesgue measure. The theorem proved in Tetunashvili [5] is the following.

Theorem 4 *Suppose given a set $E \in U(T^{d-1} \times \Phi)$, i.e., $E = \{(x_1, ..., x_d) : x_d \in A$ and $E_{(x_d)} \in U(T^{d-1})\}, A \in U(\Phi)$, and a finite-valued function $f(x_1, ..., x_d)$ on $[0,1]^d$. In addition suppose that for each $x_d \in A$ the function $f(x_1, ..., x_{d-1}, x_d)$ is μ_{d-1}-measurable on $[0,1]^{d-1}$. If*

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1,...,n_d} \prod_{j=1}^{d-1} t_{n_j}(x_j)\varphi_{n_d}(x_d) = f(x_1, ..., x_d)$$

for $(x_1, ..., x_d) \in E$, then for each $(n_1, ..., n_{d-1}) \in Z_0^{d-1}$

$$\sum_{n_d=0}^{\infty} a_{n_1,...,n_{d-1},n_d}\varphi_{n_d}(x_d) = b_{n_1,...,n_{d-1}}(x_d) < \infty, \qquad x_d \in A,$$

and for any fixed $x_d \in A$

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_{d-1}=0}^{\infty} b_{n_1,\dots,n_{d-1}}(x_d) \prod_{j=1}^{d-1} t_{n_j}(x_j) = f(x_1, \dots, x_{d-1}, x_d)$$

for $(x_1, \dots, x_{d-1}) \in E_{(x_d)}$.

Note that the first sum above is rectangular d-dimensional and the last sum is rectangular $(d-1)$-dimensional, having been already summed inside on the d-th coordinate.

4. Remarks

It seems that iterated convergence is actually equivalent to a weakened form of rectangular convergence. We use dimension two to illustrate and will call this half-rectangular convergence. A two-dimensional trig series converges half-rectangularly if, for each $X = (x, y)$, there is a non-decreasing function $g_X : Z_0 \to Z_0$ so that $\lim_{\|K\| \to \infty} S_{kj}(X)$ exists everywhere with the restriction $j \geq g_X(k)$. For example, if every g_X is the identity map then the rectangular partial sums used are "longer" in the j direction than the k direction. One can then still get the desired version of Lemma 2 and get iteration in the $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}$ direction. Conversely, if a two-dimensional trig series has iterated convergence in the $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}$ direction, it is easy to come up with the g_X to make $\sum_{m=-k}^{k} \sum_{n=-j}^{j} a_{mn} e^{i(ny+mx)}$ close to $\sum_{m=-k}^{k} (\sum_{n=-\infty}^{\infty} a_{mn} e^{iny}) e^{imx}$, thereby forcing half-rectangular convergence.

We use an example communicated to the author by J. Marshall Ash. It is easy to see that the series $\sum_{m=-\infty}^{\infty} \sqrt{m} \sin(mx)(\cos^m(y) - \cos^{3m}(y))$ converges for all x and y since it is essentially geometric in $\cos(y)$ unless $|\cos(y)| = 1$ in which case the sum is trivially zero. The powers of $\cos(y)$ can be expanded using terms involving $\cos(ny)$ where $0 \leq n \leq 3m$. When this series is then converted to exponential form we get the iterated series $\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn} e^{i(ny+mx)}$ where $a_{mn} = 0$ if $|n| > 3|m|$. This is then half-rectangular convergence using $g_X(m) = 3m$. An estimate of the coefficients shows that a_{m0} does not approach zero as m increases, so the series cannot be iterated in the reverse order.

References

[1] J.M. Ash, C. Freiling, D. Rinne, Uniqueness of rectangularly convergent trigonometric series, *Annals of Mathematics*, **137** (1993), 145–166.

[2] J.M. Ash, G. Welland, Convergence, uniqueness, and summability of multiple trigonometric series, *Trans. of the AMS*, **163** (1972), 401–436.

[3] G. Cantor, Beweis, das eine für jeden reellen Wert von x durch eine trigonometrische Reihe gegebene Funktion $f(x)$ sich nur auf eine einzige Weise in dieser Form darstellen läst, Crelles J. *für Math.*, **72** (1870), 139–142.

[4] P.J. Cohen, Topics in the theory of uniqueness of trigonometric series, *Thesis*, University of Chicago, IL, 1958.

[5] SH.T. Tetunashvili, On some multiple function series and the solution of the uniqueness problem for Pringsheim convergence of multiple trigonometric series, *Mat. Sb.* **182** (1991) No. 8; English transl. in Math. USSR Sb. **73** (1992) No. 2, 517–534.