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REPAIRING THE PROOF OF A CLASSICAL DIFFERENTIATION RESULT

Throughout E will denote a Lebesgue measurable subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, a measurable function. Theorem 4.30 on page 78 of Volume 2 of Zygmund's book [6] asserts that if $f(x+h) - 2f(x) + f(x-h) = O(h^2)$ for every $x \in E$, then f has a second Peano derivative almost everywhere in E. A more general assertion by Marcinkiewicz and Zygmund states that if f is k Riemann bounded on E, then it is k times Peano differentiable almost everywhere on E. (See Theorem 1, [2].) Superseding both of these results is a statement by Ash which claims that if f is generalized bounded of order k, then f is k times Peano differentiable everywhere in E. (See Theorem 1 in [1].) The proof of each of these assertions as well as the proof of Lemma 1, page 24 in [3] assumes certain sets to be measurable. Specifically in the case of Theorem 4.30 in [6] the sets in question are the sets

$$E_j = \{x \in E : \left|\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}\right| < j \text{ for all } 0 < |h| < \tfrac{1}{j}\}$$

for $j \in \mathbb{N}$. However, as was pointed out by Stein and Zygmund in [4], measurability of these sets is not automatic. On the other hand as a consequence of the conclusions of the assertions, the function f must be continuous a.e. on E and from that fact the measurability of the sets, E_j, follows easily.

In this paper we present a technique for fixing the proofs of all of these theorems. The procedure doesn't prove the measurability of the sets, E_j, but rather avoids the measurability question entirely. It turns out that our method is similar to that used by Stein and Zygmund in [4] but their work doesn't include what is done here nor is their result a consequence of ours. Since each of the mentioned results follows from the work of Ash in [1], we repair the

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proofs of Lemmas 2 and 3 of that article. In fact Theorems 3 and 2 of this paper are exactly the same as Lemmas 2 and 3 in [1].

Before proving these two theorems we point out that the measurability of the function f alone doesn't imply measurability of sets E_j. We gratefully acknowledge the help of Professors Krzysztof Ciesielski and Chris Freiling with this example.

Example 1 *There is a measurable function for which the corresponding set E_2 is not measurable.*

First choose two numbers $a \in (0, \frac{1}{2})$ and $\delta > 0$ such that $a + \delta < \frac{1}{2}$ and $(a - 2\delta)^2 \geq \frac{1}{8}$. (For example $a = \frac{7}{16}$ and $\delta = \frac{1}{32}$ will suffice.) By transfinite induction one can construct a set $S \subset [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ of measure 0 such that $\frac{(S+S)}{2}$ is not measurable. (See [5]. In the preliminary version, the details appears on page 136.) Let $f = \frac{1}{4}\chi_{(S-a)\cup(S+a)}$, where χ_E denotes the characteristic function of E. Since S has measure 0, f is measurable. If $x \in \frac{(S+S)}{2}$, then $x = \dfrac{s_1 + s_2}{2}$ where $s_1, s_2 \in S$ with $s_1 \leq s_2$. Let $h = x - s_1 + a = s_2 - x + a$. Then $x - h \in S - a$ and $x + h \in S + a$. Since $a + \delta < \frac{1}{2}$, it follows that $\left|\dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}\right| \geq 2$. On the other hand if $x \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$, and if there is an $h > 0$ with either $x - h \in S - a$ or $x + h \in S + a$ but not both, then since $(a - 2\delta)^2 \geq \frac{1}{8}$, it can be seen that $\left|\dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}\right| < 2$. It follows that

$$\{x \in [\tfrac{1}{2} - \delta, \tfrac{1}{2} + \delta] : |f(x+h) - 2f(x) + f(x-h)| \geq 2 \text{ for some } 0 < h < \tfrac{1}{2}\}$$
$$= \frac{(S+S)}{2}.$$

Since $\frac{(S+S)}{2}$ is not measurable, it can easily be concluded that E_2 is not measurable.

In what follows the Lebesgue outer measure of a set B is denoted by $m^*(B)$ or $m(B)$ if B is measurable.

Lemma 1 *Let 0 be a point of outer density of E, let $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$ and let $\epsilon > 0$. For each $u > 0$ set $B_u = \{v \in [u, 2u] : \alpha u + \beta v \in E\}$. Then there is a $\delta > 0$ such that if $0 < u < \delta$, then $m^*(B_u) > u(1 - \epsilon)$.*

PROOF: Let G be a G_δ cover of E. Then

$$m^*(B_u) = \int_u^{2u} \chi_G(\alpha u + \beta v)\, dv = \frac{1}{\beta} \int_{(\alpha+\beta)u}^{(\alpha+2\beta)u} \chi_G(s)\, ds.$$

Since 0 is a point of density of G,

$$\lim_{u\to 0}\int_{(\alpha+\beta)u}^{(\alpha+2\beta)u} \chi_G(s)\,ds = \frac{1}{\beta}[(\alpha+2\beta)u-(\alpha+\beta)u] = u.$$

Hence the desired $\delta > 0$ exists. □

Conforming with the notation used in [1], let $n \in \mathbb{N}$, let $\{a_0, a_1, \ldots, a_n\}$ be a set of distinct numbers and let $\{A_0, A_1, \ldots, A_n\} \subset \mathbb{R}$. There are conditions relating these two sets of numbers, but they play no role here other than to be able to assume that say $a_0 \neq 0$ and $A_0 \neq 0$.

Theorem 2 *Suppose* $|\sum_{i=0}^{n} A_i f(x+a_i t)| = O(1)$ *for all* $x \in E$. *Then* f *is bounded in a neighborhood of almost every point* $x \in E$.

PROOF: For each $j \in \mathbb{N}$ let

$$E_j = \left\{x \in E : \left|\sum_{i=0}^{n} A_i f(x+a_i t)\right| < j \text{ for all } 0 < |t| < \tfrac{1}{j}\right\}$$

and $F_j = \{x \in \mathbb{R} : |f(x)| < j\}$. Since $\cup_{j\in\mathbb{N}}(E_j \cap F_j) = E$, it suffices to show that f is bounded on some neighborhood of every point of outer density of $E_j \cap F_j$. Assume 0 is a point of outer density of $E_j \cap F_j$. Let $u > 0$. Set $B = \{v \in [u, 2u] : v \in E_j \cap F_j\}$ and for $i = 1, 2, \ldots, n$ let $C_i = \{v \in [u, 2u] : v + a_i \frac{u-v}{a_0} \in F_j\}$. The set B need not be measurable, but since f is measurable, each C_i is measurable. Also 0 is a point of outer density of B and a point of density of each C_i. By Lemma 1 there is $0 < \delta < \frac{|a_0|}{j}$ such that if $0 < u < \delta$, then $m^*(B) > \frac{u}{2}$ and $m(C_i) > u(1 - \frac{1}{2n})$ for each $i = 1, 2, \ldots, n$. Set $C = \cap_{i=1}^{n} C_i$. Then $m(C) > \frac{u}{2}$. Let $0 < u < \delta$. Then

$$\frac{u}{2} < m^*(B) \leq m^*(B\cap C) + m^*(B \setminus C) \leq$$

$$\leq m^*(B\cap C) + m([u,2u]\setminus C) \leq m^*(B\cap C) + \frac{u}{2}.$$

Thus $0 < m^*(B \cap C)$. Let $v \in B \cap C$. Then

$$\left|\sum_{i=0}^{n} A_i f(v+a_i t)\right| < j \text{ for all } 0 < |t| < \tfrac{1}{j}. \tag{1}$$

Choose $t = \dfrac{u-v}{a_0}$. Then $|t| < \frac{1}{j}$ and (1) becomes

$$\left|A_0 f(u) + \sum_{i=1}^{n} A_i f(v+a_i t)\right| < j. \tag{2}$$

Since for each $i = 1, 2, \ldots, n$ we have $v \in C_i$, from (2) it follows that $|f(u)| < \frac{j + j\sum_{i=1}^{n} |A_i|}{|A_0|}$. Therefore f is bounded in a right hand neighborhood of 0. Similarly f is bounded in a left hand neighborhood of 0. □

Theorem 3 (The sliding lemma) *Let $\alpha \geq 0$. Suppose $\sum_{i=0}^{n} A_i f(x + a_i t) = O(t^\alpha)$ for all $x \in E$. Let $a \in \mathbb{R}$. Then $\sum_{i=0}^{n} A_i f(x + (a_i - a)t) = O(t^\alpha)$ for almost every $x \in E$.*

If "O" is replaced by "o" in the hypothesis and in the conclusion, then the resulting assertion is also true.

PROOF: For each $j \in \mathbb{N}$ let

$$E_j = \{x \in E : \left|\sum_{i=0}^{n} A_i f(x + a_i t)\right| \leq j|t|^\alpha \text{ if } |t| < \tfrac{1}{j}\}.$$

Since $E = \cup_{j \in \mathbb{N}} E_j$, to prove the theorem it suffices to prove

$$\sum_{i=0}^{n} A_i f(x + (a_i - a)t) = O(t^\alpha)$$

for every point of outer density one of E_j. To simplify the notation, suppose 0 is a point of outer density of E_j. Let $t \in (0, \frac{1}{2j})$. (The case $t \in (-\frac{1}{2j}, 0)$ is proved similarly.) Let $B_i^* = \{v \in [t, 2t] : (a_i - a)t - a_0 v \in E_j\}$ and let $C_i^* = \{v \in [t, 2t] : -at + (a_i - a_0)v \in E_j\}$. Set

$$B_i = \{v \in [t, 2t] : \left|\sum_{k=0}^{n} A_k f([(a_i - a)t - a_0 v] + a_k v)\right| \leq j2^\alpha |t|^\alpha\}$$

$$C_i = \{v \in [t, 2t] : \left|\sum_{k=0}^{n} A_k f([-at + (a_i - a_0)v] + a_k t)\right| \leq j2^\alpha |t|^\alpha\}.$$

Note that B_i and C_i are measurable sets. Let $0 < \epsilon < \frac{1}{2n+1}$. By Lemma 1 there is $0 < \delta < \frac{1}{2j}$ such that $t < \delta$ implies $m^*(B_i^*) > t(1-\epsilon)$ and $m^*(C_i^*) > t(1-\epsilon)$. Using that $v < \frac{1}{j}$ it follows that $B_i^* \subset B_i$. Similarly but using that $t < \frac{1}{j}$ it is easy to show that $C_i^* \subset C_i$. Thus $m(B_i) > t(1 - \epsilon)$ and $m(C_i) > t(1 - \epsilon)$. Therefore

$$m(\cap_{i=0}^{n}(B_i \cap C_i)) > t(1 - (2n + 1)\epsilon) > 0.$$

Let $v \in \cap_i (B_i \cap C_i)$. Then

$$\left|A_0 \sum_{i=0}^{n} A_i f((a_i - a)t)\right| = \left|A_0 \sum_{i=0}^{n} A_i f([-at + (a_0 - a_0)v] + a_i t)\right| =$$

$$\Bigg|\sum_{k=0}^{n} A_k \sum_{i=0}^{n} A_i f([-at + (a_k - a_0)v] + a_i t) -$$

$$\sum_{k=1}^{n} A_k \sum_{i=0}^{n} A_i f([-at + (a_k - a_0)v] + a_i t)\Bigg| \leq$$

$$\Bigg|\sum_{i=0}^{n} A_i \sum_{k=0}^{n} A_k f([-at + (a_k - a_0)v] + a_i t)\Bigg| +$$

$$\Bigg|\sum_{i=1}^{n} A_i \sum_{k=0}^{n} A_k f([-at + (a_i - a_0)v] + a_k t)\Bigg| =$$

$$\Bigg|\sum_{i=0}^{n} A_i \sum_{k=0}^{n} A_k f([(a_i - a)t - a_0 v] + a_k v)\Bigg| +$$

$$\Bigg|\sum_{i=1}^{n} A_i \sum_{k=0}^{n} A_k f([-at + (a_i - a_0)v] + a_k t)\Bigg| \leq$$

$$2^{\alpha+1} j |t|^{\alpha} \sum_{i=0}^{n} |A_i| = M_1 |t|^{\alpha}.$$

Dividing by $|A_0|$, we have $|\sum_{i=0}^{n} A_i f((a_i - a)t)| \leq M|t|^{\alpha}$ where the constant M depends on α, j, the numbers A_i, $i = 1, 2, \ldots, n$ but not on t.

The o case is proved in a very similar manner. □

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