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Chris Freiling,* Math Dept., California State University, San Bernardino, CA, 92407, cfreilin@wiley.csusb.edu

AN OPEN SET WITH INTERMEDIATE YET SMOOTH MEASURE

Abstract

Let $\lambda(A)$ denote the Lebesgue measure of A and let $\Delta(A, I)$ denote the relative measure $\lambda(A \cap I)/\lambda(I)$.

We construct two disjoint open sets $A, B \subset [0,1]$ each having measure 1/2 such that $\lim_{h \to 0} \Delta(A, (x, x+h)) - \Delta(A, (x-h, x)) = 0$ (in fact uniformly in x). This says that the cumulative measure function $F(x) = \lambda(A \cap (0, x))$ is "uniformly smooth", meaning that $F(x+h) - 2F(x) + F(x-h) = o(h)$ (uniformly in x).

Our construction turns out to be both a modification and combination of ideas of J. P. Kahane and G. Piranian which appear in [5], where a proof of the existence of a uniformly smooth Cantor function is outlined.

The present question arose in connection with the following density theorem of O'Malley:

Theorem 1 *(O'Malley [3]) If A is a nonempty and bounded F_σ – set with right density one at each of its points, then there is a point in the complement of A where A has left density one.*

O'Malley raised the question as to whether the restriction to F_σ-sets was necessary and a prize of \$60 was offered (see [4]). The question was answered in [1] where it was shown that the theorem held for $G_{\delta\sigma}$-sets but failed for $F_{\sigma\delta}$-sets (and the prize was immediately paid!).

To answer O'Malley's question, a bounded set A was constructed so that for each x, A has left density one at x implies that A has right density one at x. The $F_{\sigma\delta}$-counterexample to O'Malley's Theorem is then created by adjoining to A all the points at which A has left density one. The question was naturally raised as to whether the set A could be modified to work for densities other

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than one, and the \$60 was re-invested. The example we create here answers that question.

If we adjoin to our set A all of the points at which A has lower left density at least β, then the new set has upper (in fact lower) right density at least β at all of its points but at no point in the complement does it have lower left density at least β. This shows that the first inequality in the following theorem must remain strict.

Theorem 2 *(Maly, Preiss, Zaijek [2]) If A is a measurable, nonempty, and bounded set with upper right density $> \beta$ at each of its points ($\beta \in (0,1)$), then there is a point in the complement of A for which A has lower left density $\geq \beta$.*

The example we create is also related to monotonicity questions involving preponderant symmetric derivatives. For example, if we let

$$AB(x) = \{h > 0 | x - h \in A,\, x + h \in B\}$$

then $AB(x)$ will always turn out to have upper density $\leq 1/2$ about $h = 0$. This naturally raises some questions. Therefore, following tradition we will (after some profit taking) reinvest the money as follows:

Question 1 *(\$10) Are there disjoint open sets A and B (perhaps the present ones) which are contained in (0,1) such that $A \cup B$ has measure one and such that for each x, $AB(x)$ has upper density strictly less than $1/2$?*

Question 2 *(\$10) Same as above with upper density $< 1/4$?*

Question 3 *\$10) Same as above with upper density $<$ some number which is $< 1/4$?*

Question 4 *(\$10) Is there a continuous function which is not monotone such that for each x, the set $\{h > 0 | f(x+h) \leq f(x-h)\}$ always has upper density less than $1/2$?*

Question 5 *(\$10) Is there a continuous function which is not constant such that for each x, the set $\{h > 0 | f(x+h) \neq f(x-h)\}$ always has upper density less than $1/2$?*

We now proceed with the construction. We will use a non-increasing sequence $\{\delta_n\} = \{1/4, 1/4, ..., 1/8, 1/8, ...1/16, 1/16, ..., ...\}$. The length of the constant subsequences is to be decided later. Note that for each n,

$$\delta_{n+1} = \delta_n \text{ or } \delta_n/2. \tag{1}$$

We will use the sequence to construct open sets $A, B \subset [0,1]$. We first define a function f which maps each open interval of the form $(a/4^n, (a+1)/4^n)$ (where a is an integer in $[-1, 4^n]$ and n is a natural number which we call the "stage" of the interval) to a binary rational between 0 and 1. If $a = -1$ or $a = 4^n$ then the interval is not in [0,1]. In this case, we map the interval to the number 1/2. For intervals in (0,1) we proceed in stages as follows:

Stage 0: The only interval for $n = 0$ is (0,1). We let f map (0,1)to 1/2.

Stage $n+1$: Assume the intervals at stage n have been assigned. We partition each "parent" interval $(a/4^n, (a+1)/4^n)$, $0 \leq a \leq 4^n - 1$ into four equally sized intervals $I < J < K < L$. The intervals $(a-1)/4^n, a/4^n)$, $(a/4^n, (a+1)/4^n)$, and $(a+1)/4^n, (a+2)/4^n)$ will have been already assigned. Call their values p, m, q respectively. If $m = 0$ or 1 then all of the four subintervals I, J, K, L, also get the value m. Otherwise, each of the subintervals gets one of the values m, $m - \delta_n$, or $m + \delta_n$ as follows:

- If $m - \delta_n > p$ then $f(I) = m - \delta_n$ and $f(J) = m + \delta_n$.
- If $m + \delta_n < p$ then $f(I) = m + \delta_n$ and $f(J) = m - \delta_n$.
- Otherwise (ie. $p \in [m - \delta_n, m + \delta_n]$), then except as noted below, $f(I) = f(J) = m$.

Similarly;

- If $m - \delta_n > q$ then $f(K) = m - \delta_n$ and $f(L) = m + \delta_n$.
- If $m + \delta_n < q$ then $f(K) = m + \delta_n$ and $f(L) = m - \delta_n$.
- Otherwise (ie. $q \in [m - \delta_n, m + \delta_n]$), then except as noted below, $f(K) = f(L) = m$.

However;

If both p and q are in $[m - \delta_n, m + \delta_n]$ then instead of making $f(I) = f(J) = f(K) = f(L) = m$, we let $f(I) = f(L) = m - \delta_n$ and $f(J) = f(K) = m + \delta_n$.

Note then, that in all cases;

At least one of the subintervals gets the value $m - \delta_n$ and at least one gets the value $m + \delta_n$ and the average of all four of them is exactly m. (2)

This completes the definition of f.

Let $d(n)$ denote the maximum difference of the function f on two neighboring intervals, both of the same size $1/4^n$. Then $d(0) = 0$, $d(1) = 2\delta_0$, and $d(n+1) \leq \max\{2\delta_n,\ d(n) - 2\delta_n,\ \delta_n\}$. The last estimate comes from considering the three cases 1)where the intervals came from the same "parent" interval at stage n or came from different "parents" whose function values differed *by* $\leq 2\delta_n$ and 2) came from different parents at stage n which differed by more than $2\delta_n$ and 3) the intervals share 0 or 1 as a common border point (in this case, the interval outside of [0,1] is mapped to 1/2 and the one inside of [0,1] is mapped to either $1/2$ or $1/2 + \delta_n$ or $1/2 - \delta_n$). By using property (1) inductively, we can then simplify the estimate:

$$d(n+1) \leq 2\delta_n \tag{3}$$

We now define the set A_n to be the union of all intervals which were assigned to 1 at some stage $\leq n$ and let B_n be the union of all intervals assigned to 0 at stage $\leq n$. Finally, let $A = \cup A_n$ and $B = \cup B_n$.

If the sequence $\{\delta_n\}$ took on a constant value δ for all $n \geq N$ then by (*) we would have that for any $n \geq N$ and any interval I of stage n which is not in B, there is a subinterval J of stage $m \leq n + 1/\delta$ such that $f(J) = 1$. Then $\lambda(A_m \cap I) \geq |J| \geq |I| \times (1/4)^{1/\delta}$. Thus the measure of the complement of $A_n \cup B_n$ approaches 0 (in a geometric fashion) as $n \to \infty$.

We would like however that our sequence $\{\delta_n\} \to 0$. Therefore, for each p, we let δ_n take on the constant value $1/2^p$ long enough so that at the stage n where it switches to $1/2^{p+1}$ we have $\lambda(A_n \cup B_n) \geq 1 - 1/2^p$. This still forces $\lambda(A \cup B) = 1$.

Next, we show that the function f calculates the relative measure of A.

Proposition 1 *For each interval I of the form $(a/4^n, (a+1)/4^n)$ where a is an integer in $[0, 4^n - 1]$ and n is a natural number, $f(I) = \Delta(A, I)$.*

PROOF. Given such an interval I and an $\epsilon > 0$, choose a stage n such that $\Delta(A_n \cup B_n, I) > 1 - \epsilon$. Then by (2), $f(I)$ is the average of f on the stage n subintervals of I. Considering the two extreme cases where the intervals not in $A_n \cup B_n$ all have value zero or all have value one, we get both that

$\Delta(A_n, I) \leq f(I) \leq \Delta(A_n, I) + \epsilon$.

and

$\Delta(A_n, I) \leq \Delta(A, I) \leq \Delta(A_n, I) + \epsilon$.

Therefore, $f(I)$ is within ϵ of $\Delta(A, I)$ and since ϵ is arbitrary, $f(I) = \Delta(A, I)$. □

Now we show the uniform smoothness:

Proposition 2 *For each $\epsilon > 0$, there is a $\delta > 0$ such that for each x and each $h < \delta$, if the intervals $[x-h, x]$ and $[x, x+h]$ both lie in [0,1] then $\Delta(A, [x, x+h])$ and $\Delta(A, [x-h, x])$ differ by at most ϵ.*

PROOF. Let $\epsilon' > 0$ be given and let $\epsilon = \min(\epsilon'/20, 1/4)$. Using (3) and the fact that $\delta_n \to 0$, choose h so small that any two neighboring intervals of the same size $< h$ have f values which differ by less than ϵ^2. Choose any x in $[h, 1-h]$. Choose n such that the number of stage n intervals which intersect $[x-h, x+h]$ is between $1/\epsilon$ and $4/\epsilon$ (inclusive). Let m be the smallest function value of one of these intervals. Then all of these intervals have a function value between m and $m + 4/\epsilon \times \epsilon^2 = m + 4\epsilon$. Furthermore, since all but two of them lie entirely inside of of $[x-h, x+h]$, the size of each one must be less than $2h/(1/\epsilon - 2)$ which is less than $4h\epsilon$ (since $\epsilon < 1/4$).

Restricting our attention now to $[x, x+h]$ the average value of the stage n intervals inside of $[x, x+h]$ is between m and $m + 4\epsilon$, and by Proposition 1, these values are the relative measure of A inside of each. Since there may be up to two pieces of a stage n interval on each end of $[x, x+h]$ which have combined relative measure less than $2 \times 4h\epsilon/h = 8\epsilon$, we have;

$m - 8\epsilon < \Delta(A, [x, x+h]) < m + 4\epsilon + 8\epsilon.$

Similarly,

$m - 8\epsilon < \Delta(A, [x-h, x]) < m + 4\epsilon + 8\epsilon.$

Therefore $\Delta(A, [x, x+h])$ and $\Delta(A, [x-h, x])$ differ by at most $20\epsilon \leq \epsilon'$. □

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