Real Analysis Exchange
Vol. 19(2), 1993/94, pp. 608–611

Chris Bernhardt , Department of Mathematics and Computer Science, Fairfield University, Fairfield, CT, 06439, cbernhardt@fair1.fairfield.edu

Ethan Coven, Department of Mathematics, Wesleyan University, Middletown, CT, 06459, ecoven@eagle.wesleyan.edu

ON THE EXISTENCE OF FUNDAMENTAL REPRESENTATIVES OF CYCLIC PERMUTATIONS IN MAPS OF AN INTERVAL

Abstract

If a cyclic permutation which is not a double has a representative in a piecewise weakly monotone map of an interval, then it has a fundamental representative in that map.

Let π be a cyclic permutation of $\{1, \ldots, n\}$, and let f be a continuous map of a compact interval to itself, a *map of an interval* for short. A *representative* π in f is a set $P = \{p_1, \ldots, p_n\}$ such that if P is labeled so that $p_1 < \cdots < p_n$, then each $f(p_i) = p_{\pi(i)}$. S. Baldwin's forcing relation [0] on cyclic permutations (an extension of Sharkovskiĭ's Theorem) is defined in terms of representatives. π *forces* θ if every map of an interval which has a representative of π also has a representative of θ. (π and θ may be permutations of different numbers of elements.)

In [0] the authors developed an efficient (polynomial-time) algorithm for deciding the forcing relation. An important ingredient of that algorithm is the notion of *fundamental representative*, defined below. We showed that to decide whether π forces θ, it is sufficient to consider only maps of an interval known as truncated horseshoe maps, and that any cyclic permutation which has a representative in a truncated horseshoe map also has a fundamental representative in it. This leads to the question: for which maps f of an interval

Key Words: interval maps, Markov graph, forcing relation

Mathematical Reviews subject classification: Primary: 58F03: Secondary: 58F20, 58F08

Received by the editors September 7, 1993

and which cyclic permutations π is it true that if π has a representative in f, then it also has a fundamental representative in f?

Informally, a representative P of π in f is fundamental if the monotonicity of f on certain intervals containing points of P is the same as the monotonicity of L_π on the corresponding intervals containing points of $\{1,\dots,n\}$, where $L_\pi : [1,n] \to [1,n]$ is obtained by "connecting the dots" of $\{(i,\pi(i)) : i = 1,\dots,n\}$ with straight lines. In order to make the definition of fundamental precise, we introduce some additional concepts and terminology.

Let $g : I \to I$ be a map of an interval and let $J_1,\dots,J_m$ be nondegenerate, compact subintervals of I such that each $\max J_k \le \min J_{k+1}$. The *Markov graph* of g with respect to $(J_1,\dots,J_m)$ is the directed graph with vertices $J_1,\dots,J_m$ and an edge $J_k \to J_\ell$ if and only if $g(J_k) \supseteq J_\ell$. The Markov graph of π is the Markov graph of L_π with respect to $([1,2],\dots,[n-1,n])$. (It may be defined in terms of π alone.)

For $\delta > 0$ small enough and for $k = 0,\dots,n$, there is a unique integer λ_k such that $L_\pi^k[1,1+\delta] \subseteq [\lambda_k,\lambda_k+1]$. The loop

$$[\lambda_0,\lambda_0+1] \to [\lambda_1,\lambda_1+1] \to \cdots \to [\lambda_{n-1},\lambda_{n-1}+1] \to [\lambda_n,\lambda_n+1]$$

in the Markov graph of π is called the *fundamental loop* of π. (Since $\lambda_0 = \lambda_n = 1$, it really is a loop, not just a path.)

A representative P of π in f is *fundamental* if for $\delta > 0$ small enough and for $k = 0,\dots,n-1$, f is *weakly monotone* , i.e., nondecreasing or nonincreasing, but not constant on $f^k[\min P, \min P+\delta]$, nondecreasing or nonincreasing according to whether L_π is increasing or decreasing on $[\lambda_k,\lambda_k + 1]$.

Finally, π is a *double* if n is even and for $k = 1,\dots,n/2$, $\pi(2k-1)$ and $\pi(2k)$ are consecutive integers. (In this case, π is a "double" of the permutation of $\{1,\dots,n/2\}$ defined by $i \mapsto j$ if $\{\pi(2i-1),\pi(2i)\} = \{2j-1,2j\}$.)

Theorem 1 *Suppose that f is a piecewise weakly monotone map of an interval and that π is not a double. If π has a representative in f, then it has a fundamental representative in f.*

PROOF. We may assume that $n \ge 2$. Let $P = \{p_1,\dots,p_n\}$ with $p_1 < \cdots < p_n$ be a representative of π in f. Let $g : [p_1,p_n] \to [p_1,p_n]$ be a continuous piecewise weakly monotone map such that every fundamental representative of π in g also is a fundamental representative of π in f. To form g, first we "truncate" f to form $f_P : [p_1,p_n] \to [p_1,p_n]$. On $[p_k,p_{k+1}]$, let

$$f_P(x) = \begin{cases} \min\bigl(f(p_k),f(p_{k+1})\bigr) & f(x) < \min\bigl(f(p_k),f(p_{k+1})\bigr) \\ \max\bigl(f(p_k),f(p_{k+1})\bigr) & f(x) > \max\bigl(f(p_k),f(p_{k+1})\bigr) \\ f(x) & \text{otherwise.} \end{cases}$$

Then let $g : [p_1, p_n] \to [p_1, p_n]$ be obtained from f_P by "pouring water" [0] into the graph of f_P on the intervals $[p_1, p_2], \ldots, [p_{n-1}, p_n]$. On $[p_k, p_{k+1}]$, let

$$g(x) = \min\big(\max_{p_k \leq y \leq x} f_P(y), \max_{x \leq y \leq p_{k+1}} f_P(y) \big).$$

Then $g = f$ on P, so P is a representative of π in g. Furthermore, g is weakly monotone but not constant on each $[p_k, p_{k+1}]$, nondecreasing or nonincreasing according to whether L_π is increasing or decreasing on $[k, k+1]$. Hence a representative of π which is fundamental in g is also fundamental in f. So we show that π has a fundamental representative in g.

The map $[k, k+1] \mapsto [p_k, p_{k+1}]$ is an isomorphism of the Markov graph of π onto the Markov graph of g with respect to $([p_1, p_2], \ldots, [p_{n-1}, p_n])$. Therefore, corresponding to the fundamental loop of π, there is a loop

$$[p_{\lambda_0}, p_{\lambda_0+1}] \to [p_{\lambda_1}, p_{\lambda_1+1}] \to \cdots \to [p_{\lambda_{n-1}}, p_{\lambda_{n-1}+1}] \to [p_{\lambda_0}, p_{\lambda_0+1}]$$

in the Markov graph of g with respect to $([p_1, p_2], \ldots, [p_{n-1}, p_n])$.

Then $J = \bigcap_{k=0}^{n} g^{-k}[p_{\lambda_k}, p_{\lambda_k+1}]$ is an interval and g^n is weakly monotone but not constant on J. Since $g^n(J) \supseteq J$, $g^n(z) = z$ for some $z \in J$. The orbit $\{z, \ldots, g^{n-1}(z)\}$ of any such z is a representative of π in g. Since the monotonicity of g on $[p_k, p_{k+1}]$ is the same of that of L_π on $[k, k+1]$, such an orbit is fundamental if for $\delta > 0$ small enough and $k = 0, \ldots, n-1$, g is weakly monotone but not constant on $g^k[z, z+\delta]$. Now $L_\pi^n(1) = 1 \in \bigcap_{k=0}^{n} L_\pi^{-k}[\lambda_k, \lambda_k + 1]$, so L_π^n is increasing on this interval. Therefore g^n is nondecreasing on J.

Let z be the largest fixed point of g^n in J such that $g^n(x) \leq x$ for all $x \leq z$. If $z = p_2$, then $g^n[p_1, p_2] = [p_1, p_2]$, hence for $k = 0, \ldots, n-1$, $\lambda = \lambda_{k+1}$ is the only integer such that $g[p_{\lambda_k}, p_{\lambda_k+1}] \supseteq [p_\lambda, p_{\lambda+1}]$. Then the (unique) edge which originates at$[p_{\lambda_k}, p_{\lambda_k+1}]$ terminates at $[p_{\lambda_{k+1}}, p_{\lambda_{k+1}+1}]$, and so the (unique) edge which originates at $[\lambda_k, \lambda_k + 1]$ terminates at $[\lambda_{k+1}, \lambda_{k+1} + 1]$. The fundamental loop of π has length n, and because π is not a double, the fundamental loop of π is *simple*, i.e., does not consist of multiple repetitions of a shorter loop [0], Lemma 3.6. But there are only $n-1$ vertices in the Markov graph of π, so $z \neq p_2$. Therefore $z < p_2$, and for $\delta > 0$ small enough, g^n is nondecreasing but not constant on $[z, z+\delta]$. Hence the orbit of z is a fundamental representative of π in f. □

If f is not piecewise weakly monotone, the concept of a representative of π being fundamental in f need not make sense. If π is a double, the proof breaks down. For then the fundamental loop of π is the repetition of shorter loops, and we conclude not that g has a representative of π, but rather that it has a representative of a cyclic permutation of fewer than n points. In fact, the result need not be true when π is a double. Let $\pi = (1324)$ – a double of

(12) – and let $f : [1,4] \to [1,4]$ be the map obtained by "connecting the dots" of

$$\{(1,3),(2,4),(3-\delta,2-\delta),(3+\delta,2+\delta),(4,1)\}$$

with straight lines, where $\delta > 0$ is small. Then $\{1,2,3,4\}$ is the unique representative of π in f. But it is not fundamental. (That (12) is itself a double is irrelevant.)

References

[1] Ll. Alsedà, J. Llibre, and M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, World Scientific, River Edge NJ, 1993

[2] S. Baldwin, *Generalizations of a theorem of Sarkovskii on orbits of continuous real-valued functions*, Discrete Math, **57**, (1987), 111–127

[3] C. Bernhardt, *The ordering of permutations induced by continuous maps of the real line*, Ergodic Theory Dynamical Systems, **7**, (1987), 155–160

[4] C. Bernhardt and E. M. Coven, *A polynomial-time algorithm for deciding the forcing relation on cyclic permutations*, Contemp. Math, **135**, (1992), 85–93