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David A. Rose, Oral Roberts University, Tulsa, Oklahoma 74171
Dragan Janković*, East Central University, Ada, Oklahoma 74820

ON FUNCTIONS HAVING THE PROPERTY OF BAIRE

By replacing the σ-ideal of meager sets with an arbitrary ideal $\mathcal{I}$ of codense sets, a theorem of M. Wilhelm is generalized. In particular, it is shown that every function $f : X \to Y$ that is almost continuous (with respect to $\mathcal{I}$) and which has the property of Baire (with respect to $\mathcal{I}$) is θ-continuous.

In [1], Marek Wilhelm proved that each function $f : (X, \tau) \to (Y, \sigma)$ from a Baire space into a regular space (not necessarily T_0) is continuous if (and only if) it is almost continuous and has the property of Baire. Recall that a function has the property of Baire if its preimages of open sets have the property of Baire. A set has the property of Baire if it differs symmetrically from an open set by a meager amount. That is, $A \subseteq X$ has the property of Baire if for some open set U, $A \triangle U = (A \setminus U) \cup (U \setminus A)$ is meager. The almost continuity of Wilhelm's theorem appeared first in [2] as a strengthening of a condition studied by H. Blumberg [3] in connection with Blumberg spaces. It appeared later under its present name in [2] where some of the results of [4] were strengthened. A function is almost continuous if its preimages of open sets are almost open. A set $A \subseteq X$ is almost open if it is contained in the interior of the set of all points of X at which A is not locally meager. That is, A is almost open if $A \subseteq \text{int}_\tau A^*(\mathcal{M}(\tau))$ where $\mathcal{M}(\tau)$ is the σ-ideal of (τ-)meager subsets of X and $A^*(\mathcal{M}(\tau)) = \{x \in X \mid x \in U \in \tau \Rightarrow U \cap A \notin \mathcal{M}(\tau)\}$. A pointwise version of almost continuity exists and in one sense, almost continuity is quite weak. In particular, Bradford and Goffman showed that every real-valued function on a metric space is almost continuous at each point of a comeager set [2]. In the above definition of $A^*(\mathcal{M}(\tau))$, if $\mathcal{M}(\tau)$ is replaced with the trivial ideal $\{\emptyset\}$, we have $A^*(\{\emptyset\}) = \text{cl}_\tau A$ and clearly,

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$A^*(\mathcal{M}(\tau)) \subseteq \mathrm{cl}_\tau A$. Let us agree that a subset A of X is nearly open if $A \subseteq \mathrm{int}_\tau \mathrm{cl}_\tau A$, and that a function $f : (X, \tau) \to (Y, \sigma)$ is nearly continuous if preimages of open sets are nearly open. Then near continuity is weaker than almost continuity since every almost open set is nearly open. Near continuity was introduced in [3] for functions between Euclidean spaces and later for functions between arbitrary topological spaces in [5]. It has also been called almost continuity in [6] so that in contrast, the almost continuity of [2] has been called categorical almost continuity in [7]. Near continuity is best known for its use in the closed graph and open mapping theorems of functional analysis whereas, almost continuity has been useful in the study of Blumberg spaces and was used by Wilhelm to obtain a Souslin graph theorem. The following example shows that in Wilhelm's theorem above, almost continuity cannot be weakened to near continuity.

Example 1 *Let $\mathbb{R}$ be the usual space of real numbers. Then $\mathbb{R}$ is a regular Baire space and if $d : \mathbb{R} \to \mathbb{R}$ is the Dirichlet function taking the value 1 at each irrational and 0 at each rational, then d is both nearly continuous and has the property of Baire. Yet d is nowhere (even weakly) continuous.*

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous (θ-continuous) if for each $x \in X$, whenever $f(x) \in V \in \sigma$, there exists $U \in \tau$ with $x \in U$ such that $f(U) \subseteq \mathrm{cl}_\sigma V$ ($f(\mathrm{cl}_\tau U) \subseteq \mathrm{cl}_\sigma V$). Of course, continuity $\Rightarrow$ θ-continuity $\Rightarrow$ weak continuity and all are equivalent if (Y, σ) is regular. Thus, if Wilhelm's result were to be generalized by removing the space conditions such as regularity on the codomain, an acceptable conclusion for the function would be weak continuity or θ-continuity. However, by the example, involvement of $\mathcal{M}(\tau)$ in the almost continuity hypothesis cannot be completely avoided without making other concessions. For example if we say that a set has the ideal-property of Baire if it differs symmetrically from an open set by an ideal amount, then every function having the $\{\emptyset\}$-property of Baire is continuous with no other hypothesis needed. In this paper we will show that the Wilhelm result can be generalized by removing the space conditions on X and Y and replacing the function conditions of almost continuity and having the property of Baire by these properties relativized to an arbitrary codense ideal, i.e. one whose members are codense (have empty interior). The result of Wilhelm is the special case where the ideal is $\mathcal{M}(\tau)$ since $\mathcal{M}(\tau)$ is codense if and only if (X, τ) is a Baire space. The surprise is that in the generalization the ideal is neither required to be (τ-)local nor contain $\mathcal{N}(\tau)$, the ideal of nowhere dense subsets of (X, τ), usually considered to be two very important properties of $\mathcal{M}(\tau)$. Recall that an ideal is a nonempty family of subsets such that subsets of members are members and finite unions of members are members, and is a σ-ideal if it is closed under countable union of its members. An ideal $\mathcal{I}$ is

(τ-)local if it contains all subsets of (X, τ) which are locally in $\mathcal{I}$. A subset A is locally in $\mathcal{I}$ if it has an open cover $\mathcal{U} \subseteq \tau$ such that $U \cap A \in \mathcal{I}$, $\forall U \in \mathcal{U}$. Of course, $\mathcal{M}(\tau) = \Sigma(\mathcal{N}(\tau))$, the σ-extension of $\mathcal{N}(\tau)$ and is therefore τ-local since $\mathcal{N}(\tau)$ is (easily seen to be) τ-local [8], [9], or [10] and it is shown generally in [11] that $\Sigma(\mathcal{I})$ is local whenever $\mathcal{I}$ is local. However, the τ-locality of $\mathcal{M}(\tau)$ is commonly known as the Banach Category Theorem [12] and was proven for metric spaces in [13] and then for general topological spaces in [14]. Though precise ideal generalizations are discussed in the next section, the main result of this paper is the following generalization of Wilhelm's theorem.

Theorem *If $\mathcal{I}$ is a τ-codense ideal and $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is an $\mathcal{I}$-almost continuous function having the $\mathcal{I}$-property of Baire, then $f : (X, \tau) \to (Y, \sigma)$ is θ-continuous.*

1. Ideal Generalization and Pre-Lemma-naries

We will denote by $(X, \tau, \mathcal{I})$ a topological space (X, τ) with an ideal $\mathcal{I}$ of subsets of X.

Definition 1

a) *The ideal $\mathcal{I}$ is (τ-)codense if $\mathcal{I} \cap \tau = \{\emptyset\}$, i.e. $A \in \mathcal{I} \Rightarrow int\, A = \emptyset$.*

b) $\forall A \subseteq X$, $A^*(\tau, \mathcal{I}) = A^*(\mathcal{I}) = \{x \in X \mid \forall U \in \tau,\ x \in U \Rightarrow U \cap A \notin \mathcal{I}\}$.

c) $\forall A \subseteq X$, *A is locally in $\mathcal{I}$ if $A \cap A^*(\mathcal{I}) = \emptyset$.*

d) *The ideal $\mathcal{I}$ is (τ-)local if $A \in \mathcal{I}$ whenever A is locally in $\mathcal{I}$.*

e) *The topology $\tau^*(\mathcal{I})$ is the smallest expansion of τ for which members of $\mathcal{I}$ are closed.*

f) *The set of $\mathcal{I}$-almost open subsets of X is $AO(X, \tau, \mathcal{I}) = \{A \subseteq X \mid A \subseteq int_\tau A^*(\mathcal{I})\}$ and a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $\mathcal{I}$-almost continuous if $f^{-1}(V) \in AO(X, \tau, \mathcal{I})$ for all $V \in \sigma$. In case $\mathcal{I}$ is the trivial ideal $\{\emptyset\}$, $A^*(\mathcal{I}) = cl_\tau A$, so that $AO(x, \tau, \mathcal{I})$ is the family of nearly open subsets of the space X, denoted $NO(X, \tau)$, and also each function $f : (X, \tau) \to (Y, \sigma)$ is nearly continuous if $f^{-1}(V) \in NO(X, \tau)$ whenever $V \in \sigma$.*

g) *The family of subsets of X having the $\mathcal{I}$-property of Baire is $Br(X, \tau, \mathcal{I}) = \{B \subseteq X \mid \exists U \in \tau$ such that $B \triangle U \in \mathcal{I}\}$. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ has the $\mathcal{I}$-property of Baire if $f^{-1}(V) \in Br(X, \tau, \mathcal{I})$ for each $V \in \sigma$.*

Remarks

1) *If $A, B \subseteq X$ and $A \triangle B \in \mathcal{I}$, then $A^*(\mathcal{I}) = B^*(\mathcal{I})$. Also, $\forall A \in \mathcal{I}$, $A^*(\mathcal{I}) = \emptyset$.*

2) *The sets of the type $U - I$ where $U \in \tau$ and $I \in \mathcal{I}$ form an open basis for $\tau^*(\mathcal{I})$ and every $\tau^*(\mathcal{I})$-open set is of this type if $\mathcal{I}$ is τ-local. For each $A \subseteq X$, $cl_{\tau^*(\mathcal{I})}A = A \cup A^*(\mathcal{I})$.*

3) *As mentioned earlier, the ideals $\mathcal{N}(\tau)$ and $\mathcal{M}(\tau)$ are always τ-local and usually, $\tau^*(\mathcal{N}(\tau))$ is denoted τ^α [15], [16], [17], or [10], whereas $\tau^*(\mathcal{M}(\tau))$ is denoted τ^μ.*

4) *$\mathcal{I}$-almost openness generalizes openness precisely when $\mathcal{I}$ is codense, i.e., $\tau \subseteq AO(X, \tau, \mathcal{I})$ if and only if $\mathcal{I}$ is τ-codense. Moreover, $\mathcal{I}$ is τ-codense if and only if for all $U \in \tau$, $U^*(\mathcal{I}) = cl_\tau U$.*

The following lemmas are foundational for the proof of the Theorem which will be presented in the next section. Lemma 1 is a decomposition of openness (for sets) into near openness and local closedness. Recall that a set is locally closed if it can be expressed as an intersection of an open set with a closed set. A useful characterization of local closedness for a subset A of (X, τ) is that $(\mathrm{cl}_\tau A) - A$ is closed in (X, τ). The family of all locally closed subsets of the space (X, τ) is denoted $LC(X, \tau)$. It is also useful to recall that a set $A \in NO(X, \tau) \Leftrightarrow A = U \cap D$ for some $U \in \tau$ and for some dense subset D of (X, τ).

Lemma 1 [17] *For any topological space (X, τ), $\tau = NO(x, \tau) \cap LC(X, \tau)$.*

An immediate consequence of Lemma 1 is the decomposition of continuity into near continuity and LC-continuity found in [18] where a function $f : (X, \tau) \to (Y, \sigma)$ is LC-continuous if $f^{-1}(V) \in LC(X, \tau)$ for each $V \in \sigma$.

Our next lemma together with Remark 4 above yields a further characterization of τ-codenseness for the ideal $\mathcal{I}$, the identification of almost open sets with nearly open sets relative to the topology $\tau^*(\mathcal{I})$. The proof is omitted but can be obtained from the fact that for any ideal $\mathcal{I}$, any dense subset D of $(X, \tau^*(\mathcal{I}))$, and any $U \in \tau^*(\mathcal{I})$, $(U \cap D)^*(\mathcal{I}) = U^*(\mathcal{I})$.

Lemma 2 *If $\mathcal{I}$ is τ-codense, then $AO(X, \tau, \mathcal{I}) = NO(X, \tau^*(\mathcal{I}))$.*

The next four lemmas follow easily from known results. Lemma 3 follows from Theorems 1 and 2 jointly of [19], Lemma 4 follows from Theorem 3.1 of [20], and Lemma 5 follows from Theorem 3.5 of [20] (or Theorem 4.2 of [21]). Lemma 6 is Theorem 6.4 of [9]. The join of two ideals $\mathcal{I}$ and $\mathcal{J}$, $\mathcal{I} \vee \mathcal{J} = \{I \cup J \mid I \in \mathcal{I} \text{ and } J \in \mathcal{J}\}$ is the smallest ideal containing $\mathcal{I} \cup \mathcal{J}$, and $\mathcal{J}$ is an extension of $\mathcal{I}$ if $\mathcal{I} \subseteq \mathcal{J}$.

Lemma 3 *If the ideal $\mathcal{I}$ is τ-local and contains $\mathcal{N}(\tau)$, then $Br(X,\tau,\mathcal{I}) = LC(X,\tau^*(\mathcal{I}))$.*

Lemma 4 *If $\mathcal{I}$ is an ideal of subsets of a topological space (X,τ), $\tilde{\mathcal{I}} = \{A \subseteq X \mid A^*(\mathcal{I}) \in \mathcal{N}(\tau)\}$ is a local ideal extension of $\mathcal{I} \vee \mathcal{N}(\tau)$.*

Lemma 5 *For any ideal $\mathcal{I}$ of subsets of (X,τ), $\tilde{\mathcal{I}}$ is τ-codense if and only if $\mathcal{I}$ is τ-codense.*

Let us recall that for any topology τ on X, the semiregularization of τ is the topology $\tau_s \subseteq \tau$ generated by the regular open sets of (X,τ).

Lemma 6 *If the ideal $\mathcal{I}$ is τ-codense, then $(\tau^*(\mathcal{I}))_s = \tau_s$.*

Properties shared by all topologies on X having the same semiregularization are called semiregular properties. Our last lemma asserts that θ-continuity is a semiregular property for the domain space.

Lemma 7 *If (X,τ,ρ) is a bitopological space with $\tau_s = \rho_s$, then for all functions $f : X \to (Y,\sigma)$, $f : (X,\tau) \to (Y,\sigma)$ is θ-continuous if and only if $f : (X,\rho) \to (Y,\sigma)$ is θ-continuous.*

Before proving the Theorem in the next section we give the following example to show that without regularity on (Y,σ), a function satisfying the hypotheses need not be continuous.

Example 2 *Let $(\mathbb{R},\nu)$ be the usual space of real numbers and let $f : (\mathbb{R},\nu) \to (\mathbb{R},\nu^\alpha)$ be the identity function. Then f is nearly continuous since $\nu^\alpha \subseteq NO(\mathbb{R},\nu)$ and f has the property of Baire since each $V \in \nu^\alpha$ has the form $U - I$ for some $U \in \nu$ and for some $I \in \mathcal{N}(\nu)$ and hence $V \triangle U \in \mathcal{M}(\nu)$. Moreover, $\mathcal{I} = \mathcal{M}(\nu))$ is ν-codense since $(\mathbb{R},\nu)$ is a Baire space. Yet, f is nowhere continuous. However, $f : (\mathbb{R},\nu^\alpha) \to (\mathbb{R},\nu^\alpha)$ is continuous and hence θ-continuous so that by Lemmas 6 and 7, $f : (\mathbb{R},\nu) \to (\mathbb{R},\nu^\alpha)$ is θ-continuous.*

2. Proof of the Theorem

We now restate and prove our ideal generalization of Wilhelm's theorem.

Theorem *If $\mathcal{I}$ is a τ-codense ideal and $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is an $\mathcal{I}$-almost continuous function having the $\mathcal{I}$-property of Baire, then $f : (X,\tau) \to (Y,\sigma)$ is θ-continuous.*

PROOF. Let $\mathcal{I}$ be a τ-codense ideal and let $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ be an $\mathcal{I}$-almost continuous function having the $\mathcal{I}$-property of Baire. By Lemma 3 and the fact that $\tilde{\mathcal{I}}$ extends $\mathcal{I}$, we have $Br(X,\tau,\mathcal{I}) \subseteq Br(X,\tau,\tilde{\mathcal{I}}) = LC(X,\tau^*(\tilde{\mathcal{I}}))$. Also, by Lemma 2, $AO(X,\tau,\mathcal{I}) = NO(X,\tau^*(\mathcal{I}))$. Since $\mathcal{I}$ is τ-codense, $\mathcal{I}$ is also $\tau^*(\mathcal{I})$-codense and since each member of $\mathcal{I}$ is a closed subset of $(X,\tau^*(\mathcal{I}))$, it follows that $\mathcal{I} \subseteq \mathcal{N}(\tau^*(\mathcal{I}))$. We claim that also $\mathcal{N}(\tau) \subseteq \mathcal{N}(\tau^*(\mathcal{I}))$. For if $E \in \mathcal{N}(\tau)$ and $E = \mathrm{cl}_\tau E$, then E is closed in $(X,\tau^*(\mathcal{I}))$. But, if $E \notin \mathcal{N}(\tau^*(\mathcal{I}))$, $\exists U \in \tau$ and $\exists I \in \mathcal{I}$ with $\emptyset \neq U - I \subseteq E$. Then $\emptyset \neq U = (U \cap I) \cup (U - I) \in \mathcal{I} \vee \mathcal{N}(\tau) \subseteq \tilde{\mathcal{I}}$ contrary to the fact that $\tilde{\mathcal{I}}$ is τ-codense. So $E \in \mathcal{N}(\tau^*(\mathcal{I}))$ and therefore $\mathcal{N}(\tau) \subseteq \mathcal{N}(\tau^*(\mathcal{I}))$. Thus, $\mathcal{I} \vee \mathcal{N}(\tau) \subseteq \mathcal{N}(\tau^*(\mathcal{I}))$. Note that since $\tau \subseteq \tau^*(\mathcal{I})$ and $\mathcal{N}(\tau^*(\mathcal{I}))$ is $\tau^*(\mathcal{I})$-local, $\mathcal{N}(\tau^*(\mathcal{I}))$ is a τ-local ideal. By Corollary 3.2 of [11], $\tilde{\mathcal{I}}$ is the smallest τ-local extension of the ideal $\mathcal{I} \vee \mathcal{N}(\tau)$, i.e., the intersection of all τ-local ideals containing $\mathcal{I} \vee \mathcal{N}(\tau)$, and hence $\tilde{\mathcal{I}} \subseteq \mathcal{N}(\tau^*(\mathcal{I}))$. It is easily verified that $[\tau^*(\mathcal{I})]^*(\tilde{\mathcal{I}}) = \tau^*(\tilde{\mathcal{I}})$ so that $\tau^*(\mathcal{I}) \subseteq \tau^*(\tilde{\mathcal{I}}) \subseteq [\tau^*(\mathcal{I})]^\alpha$ since also, $[\tau^*(\mathcal{I})]^\alpha = [\tau^*(\mathcal{I})]^*(\mathcal{N}(\tau^*(\mathcal{I})))$ and $\tilde{\mathcal{I}} \subseteq \mathcal{N}(\tau^*(\mathcal{I}))$. It was shown in [15] that each topology intermediate to a base topology and the α-topology expansion of the base topology has the same α-expansion as that of the base topology so that $[\tau^*(\tilde{\mathcal{I}})]^\alpha = [\tau^*(\mathcal{I})]^\alpha$. By Theorem 3.4 of [10], two topologies σ and ρ on a set X have the same α-expansion, $\sigma^\alpha = \rho^\alpha$, if and only if for each subset $A \subseteq X$, $\mathrm{int}_\sigma \mathrm{cl}_\sigma A = \mathrm{int}_\rho \mathrm{cl}_\rho A$. It follows that $AO(X,\tau,\mathcal{I}) = NO(X,\tau^*(\mathcal{I})) = NO(X,\tau^*(\tilde{\mathcal{I}}))$. Now we have that $AO(X,\tau,\mathcal{I}) \cap Br(X,\tau,\mathcal{I}) \subseteq NO(X,\tau^*(\tilde{\mathcal{I}})) \cap LC(X,\tau^*(\tilde{\mathcal{I}})) = \tau^*(\tilde{\mathcal{I}})$ by Lemma 1, so that $f : (X,\tau^*(\tilde{\mathcal{I}})) \to (Y,\sigma)$ is θ-continuous being continuous. It follows from Lemmas 5, 6, and 7 that $f : (X,\tau) \to (Y,\sigma)$ is also θ-continuous. □

3. Some Consequences

It is easily shown that $AO(X,\tau,\mathcal{I})$ is closed under arbitrary union whereas $Br(X,\tau,\mathcal{I})$ is not in general. This suggests an improvement of the Theorem by relaxing the property of Baire for the function to the extent that only basic open sets must have preimages with the property of Baire.

Definition 2 *A function $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ has the $\mathcal{I}$-property of Baire locally if there exists an open basis β for σ such that $f^{-1}(V)$ has the property of Baire for each $V \in \beta$. Here "$\mathcal{I}$" will be suppressed if $\mathcal{I} = \mathcal{M}(\tau)$.*

Corollary 1 *If $\mathcal{I}$ is τ-codense and $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is $\mathcal{I}$-almost continuous and has the $\mathcal{I}$-property of Baire locally, then $f : (X,\tau) \to (Y,\sigma)$ is θ-continuous.*

PROOF. Let β be an open basis for σ with respect to which $f : (X,\tau,\mathcal{I}) \to (Y,\sigma)$ has the $\mathcal{I}$-property of Baire locally. If also $\mathcal{I}$ is τ-codense and f is $\mathcal{I}$-almost continuous then as in the proof of the Theorem, $f^{-1}(\beta) \subseteq AO(X,\tau,\mathcal{I}) \cap Br(X,\tau,\mathcal{I}) \subseteq \tau^*(\tilde{\mathcal{I}})$ implies $f^{-1}(\sigma) \subseteq \tau^*(\tilde{\mathcal{I}})$. □

Example 3 *Let $g : \mathbb{R} \to Z$ be the greatest integer function from the usual space of reals to the subspace of integers and let $\mathfrak{F}$ be the ideal of finite subsets of $\mathbb{R}$. Then g has the $\mathfrak{F}$-property of Baire locally but does not have the $\mathfrak{F}$-property of Baire (globally). For if $E \subseteq Z$ is the set of even integers, $g^{-1}(E)$ is an infinite (discrete) union of half open intervals which does not differ symmetrically from an open set by a finite amount.*

Note that for any space (X, τ), $B(X, \tau, \mathcal{M}(\tau))$ is a σ-field, i.e., it is closed under complementation and countable union. So every function $f : (X, \tau) \to (Y, \sigma)$ into a second countable space (Y, σ) has the property of Baire locally if and only if it has the property of Baire (globally).

For the next result, recall that a space (X, τ) is Frechet if for each $A \subseteq X$, and for each $x \in \mathrm{cl}_\tau A$, there is a sequence $\{x_n \mid n < \omega\} \subseteq A$ which converges to x. Let us also say that a space (Y, σ) is strongly locally countably compact if there is an open basis β for σ such that $\mathrm{cl}_\sigma V$ is countably compact for each $V \in \beta$. Since nonregular Hausdorff countably compact spaces exist, the continuity of f in the following corollary does not follow instantly from θ-continuity. Instead, it does follow from a decomposition of continuity into weak continuity plus local weak* continuity [22]. Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is locally weak* continuous if there exists an open basis β for σ such that $f^{-1}(FrV)$ is closed for each $V \in \beta$ where $FrV = (\mathrm{cl}_\sigma V) - V$ is the frontier of V.

Corollary 2 *If $\mathcal{I}$ is a τ-codense ideal of subsets of X, (X, τ) is Frechet, and (Y, σ) is a Hausdorff strongly locally countably compact space, then $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $\mathcal{I}$-almost continuous and has the $\mathcal{I}$-property of Baire locally if and only if $f : (X, \tau) \to (Y, \sigma)$ is continuous.*

PROOF. Only the necessity is required. From Corollary 1, f is θ-continuous and hence weakly continuous. Also the graph of f, $G(f) \subseteq X \times Y$ is closed since f is weakly continuous and (Y, σ) is Hausdorff [23]. Thus, if β is an open basis for σ with respect to which (Y, σ) is strongly locally countably compact, then for each $V \in \beta$, FrV is countably compact being a closed subspace of the countably compact set $\mathrm{cl}_\sigma V$. Therefore, $f^{-1}(FrV)$ is closed for each $V \in \beta$ since (X, τ) is Frechet and $G(f)$ is closed. The conclusion follows from the aforementioned decomposition of continuity. □

Before giving our final corollary, we remark that an example of a discontinuous function from a non-Baire space into a regular space which is almost continuous and which has the property of Baire would serve to show the need for the Baire space hypothesis in Wilhelm's result as well as the essentiality of codenseness for $\mathcal{I}$ in our Theorem.

Corollary 3 *If $(\mathbb{R},\nu)$ is the usual space of real numbers, then $f:(\mathbb{R},\nu,\mathcal{I})\to(\mathbb{R},\nu)$ is $\mathcal{I}$-almost continuous and has the $\mathcal{I}$-property of Baire locally if and only if $f:(\mathbb{R},\nu)\to(\mathbb{R},\nu)$ is continuous, for any of the following choices of $\mathcal{I}$:*

1) *$\mathcal{L}_0$, the σ-ideal of Lebesgue null sets,*

2) *$\mathfrak{F}$, the ideal of finite subsets of $\mathbb{R}$,*

3) *$\mathcal{C}$, the σ-ideal of countable subsets of $\mathbb{R}$,*

4) *$\mathcal{S}(\nu)$, the ideal of scattered subsets of $(\mathfrak{R},\nu)$,*

5) *$\mathcal{N}(\nu)$, the ideal of nowhere dense subsets of $(\mathbb{R},\nu)$,*

6) *$\mathcal{M}(\nu)$, the σ-ideal of meager subsets of $(\mathbb{R},\nu)$, and*

7) *$\mathcal{O}_1$, the σ-ideal of subsets of $\mathbb{R}$ with cardinality at most ω_1 under $ZFC+\neg CH$.*

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