Real Analysis Exchange
Vol. 19(2), 1993/94, pp. 579–588

Vasile Ene, Department of Mathematics, University-Constanta, Romania

A FUNDAMENTAL LEMMA FOR MONOTONICITY

Abstract

The paper contains several useful definitions which lead to the following monotonicity theorem: if F is $\mathcal{C}_d \cap (\overline{M})$ on $[a,b]$ and $F'(x) \leq 0$ a.e. where F is derivable, then F is decreasing on $[a,b]$.

The main result of this paper is Lemma 4, which is a generalization of Lemma 6 of [1].

The following theorem of Banach ([4], p. 286) is well known:

Theorem (Banach) *Any function which is continuous and satisfies Lusin's condition (N) on an interval, is derivable at every point of a set of positive measure.*

Of course, condition (N) implies condition T_2, and it is this fact that leads to the proof of Banach's theorem.

In [3], Foran generalizes this result, showing that Banach's theorem remains true, if condition (N) is replaced by Foran's condition (M).

An improvement of Foran's theorem is given in [1] (see Theorem 9), condition (M) is replaced by condition $(\overline{M})$.

In our article, using Lemma 4, we improve this last result in Theorem 5, replacing the continuity by condition $\mathcal{C}_d$

Theorem 5 is then used to prove a monotonicity theorem (Theorem 6), which permits an extension (see Corollary 1) of Corollary 3 of [1]. Both the Corollary of [1] and Corollary 1 extend the following theorem of Nina Bary (see [4], p.286): (condition (N) is replaced by (M) and the continuity by $\mathcal{C}_i$):

Theorem (Bary) *Every continuous function, which satisfies condition (N) and whose derivative is nonnegative at almost every point x where F is derivable, is monotone nondecreasing.*

Key Words: Darboux$_+$, $\mathcal{C}_d$, monotone$_*$, $\overline{AC}$, VB, Foran's condition (M), $(\overline{M})$

Mathematical Reviews subject classification: Primary: 26A24

Received by the editors April 23, 1993

In what follows we shall need the following results on VB^*G functions (sometimes this condition is called VBG^* or BVG^*).

Theorem 1 ([4], p.230) *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b], F \in VB^*G$ on P. Then:*

(i) F is derivable a.e. on P.

(ii) $|F(N)| = \Lambda(B(F;N)) = 0$, where $N = \{x \in P :$ $F'(x)$ does not exist finite or infinite$\}$.

Theorem 2 ([4], p.234) *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b]$. If $\overline{F}'(x) < \infty$ on P then $F \in VB^*G$ on P.*

Theorem 3 ([1], p.434) *Let $F : [a,b] \to \mathbb{R}$, and let P be an uncountable subset of $[a,b]$. If $F \in VB^*G$ on P then F is continuous n.e. on P.*

Definition 1 *Let $F : [a,b] \to \mathbb{R}$. F is said to be $\mathcal{D}_-$ on $[a,b]$, if from $a \leq \alpha < \beta \leq b$ and $F(\beta) < F(\alpha)$, it follows that $[F(\beta), F(\alpha)] \subset F([\alpha,\beta])$. Let $\mathcal{D}_+ = \{F : -F \in \mathcal{D}_-\}$. Clearly $D = \mathcal{D}_+ \cap \mathcal{D}_-$, where D is the class of Darboux functions on $[a,b]$.*

Definition 2 *Let $F : P \to \mathbb{R}$. F is said to be C_i at $x_0 \in P$, if*

$$\overline{\lim}_{\substack{x \to x_0^- \\ x \in P^0}} F(x) \leq F(x_0) \leq \underline{\lim}_{\substack{x \to x_0^+ \\ x \in P^0}} F(x)$$

Let $C_d = \{F : -F \in C_i\}$. Clearly $C = C_d \cap C_i$ on P, where C is the class of functions continuous on P.

Theorem 4 ([2], p.395) *$C_d \subset \mathcal{D}_+$ on an interval.*

Lemma 1 *Let $F : [a,b] \to \mathbb{R}$. If $\overline{\lim}_{y \to x^-} F(y) \geq F(x), x \in (a,b]$ and $F(x) \geq \overline{\lim}_{y \to x^+} F(y), x \in [a,b)$ then $F \in \mathcal{D}_+$ on $[a,b]$.*

Let $a \leq a' < b' \leq b$ such that $F(a') < F(b')$ and let $\alpha \in (F(a'), F(b'))$. Let $B = \{x \in [a',b'] : F(x) \geq \alpha\}$. Clearly $b' \in B$. Hence $B \neq \emptyset$. Let $c = \inf$ (B) and suppose that $c \notin B$. Then c is a right accumulation point of B, and

$$F(c) \geq \overline{lim}_{y \to c^+} F(y) \geq \overline{\lim}_{\substack{y \to c^+ \\ y \in B}} F(y) \geq \alpha,$$

a contradiction. It follows that $c \in B$. Since $F(c) \geq \alpha$ it follows that $c \neq a'$and $F(x) < \alpha, x \in [a',c)$. By hypothesis, $F(c) \leq \overline{\lim}_{y \to c^-} F(y) \leq \alpha$, hence $F(c) = \alpha$. It follows that $F \in \mathcal{D}_+$ on $[a,b]$.

Definition 3 *Let $F : [a,b] \to \mathbb{R}$, and let P be a subset of $[a,b], c = inf\,(P), d = sup\,(P)$. F is said to be left increasing$_*$ (respectively right increasing$_*$) on P, if $F(x_1) \leq F(x_2)$, whenever $c \leq x_1 < x_2 \leq d$ and $x_1 \in P$ (respectively $x_2 \in P$). F is said to be increasing$_*$ on P, if it is simultaneously left increasing$_*$ and right increasing$_*$ on P. In the above, if $F(x_1) < F(x_2)$, we obtain the conditions: strictly left increasing$_*$, strictly right increasing$_*$, strictly increasing$_*$. Similarly, we can define conditions: decreasing$_*$, monotone$_*$, left decreasing$_*$, etc. Clearly monotone$_*$ = monotone on an interval.*

Lemma 2 *Let $F : [a,b] \to \mathbb{R}$, $F(a) < F(b)$ such that $\overline{\lim}_{y\to x^-} F(y) \geq F(x)$, for $x \in (a,b]$ and $F(x) \geq \overline{\lim}_{y\to x^+} F(y)$, for $x \in [a,b)$.*

(i) If $b_0 = inf\,\{x : F(x) = F(b)\}$ then $F(b_0) = F(b)$;

(ii) If $x_y = inf\,\{x \in (a,b_0) : F(x) = y\}$, $y \in (F(a), F(b))$ then $F(x_y) = y$.

(iii) There exists a set $A \subset [a,b_0], a, b_0 \in A$, such that F is strictly right increasing$_*$ on A and $F(A) = [F(a), F(b)]$.

(iv) If $A_+ = \{x : x$ is a right accumulation point of $A\}$ then F is right increasing$_*$ on $A \cup A_+$ and $F(A \cup A_+) = [F(a), F(b)]$.

PROOFS.

By Lemma 1, $F \in \mathcal{D}_+$ on $[a,b]$.

(i) Suppose on the contrary that $F(b_0) \neq F(b)$. Then b_0 is a right accumulation point for the set $\{x : F(x) = F(b)\}$. Hence $F(b_0) \geq \overline{\lim}_{x\to b_0^+} F(x) \geq F(b)$. It follows that $F(b_0) > F(b)$. Since $F(a) < F(b)$ and $F \in \mathcal{D}_+$, there exists $c \in (a,b_0)$, such that $F(c) = F(b)$. This contradicts the definition of b_0.

(ii) Since $F \in \mathcal{D}_+, \{x \in (a,b_0) : F(x) = y\} \neq \emptyset$. Suppose on the contrary that $F(x_y) \neq y$. Then x_y is a right accumulation point for the set $\{x \in (a,b_0) : F(x) = y\}$. Hence $F(x_y) \geq \overline{\lim}_{x\to x_y^+} F(x) \geq y$. It follows that $F(x_y) > y$. But $F(a) < y$. Since $F \in \mathcal{D}_+$, there exists $c \in (a, x_y)$ such that $F(c) = y$. This contradicts the definition of x_y.

(iii) Let $A = \{x_y : y \in [F(a), F(b)]\}$, where x_y is defined in (ii). Clearly $a, b_0 \in A$. Since $F(x_y) = y, F(A) = [F(a), F(b)]$. Let $a \leq x_1 < x_2 \leq b_0$, $x_2 \in A$. Then $x_2 = x_{F(x_2)}$ and $F(x_1) \neq F(x_2)$. Suppose on the contrary that $F(x_2) < F(x_1)$. Since $F(x_2) > F(a)$ and $F \in \mathcal{D}_+$, there exists $c \in (a, x_1)$, such that $F(c) = F(x_2)$. This contradicts the fact that $x_2 = x_{F(x_2)}$. Thus $F(x_1) < F(x_2)$ and F is strictly right increasing$_*$ on A.

(iv) Let $x_0 \in A_+, x_0 \neq b_0$. We show that $F(x_0) = \inf\{F(x) : x \in A, x > x_0\}$. By (iii), F is strictly increasing and bounded on A. Hence the above infimum is finite and belongs to $[F(a), F(b)]$. Since F is strictly right increasing$_*$ on $A, F(x_0) < F(x)$, for each $x \in A, x > x_0$. Hence $F(x_0) \leq \inf\{F(x) : x \in A, x > x_0\}$. But

$$F(x_0) \geq \overline{\lim}_{x \to x_0^+} F(x) \geq \overline{\lim}_{\substack{x \to x_0^+ \\ x \in A}} F(x) = \inf\{F(x) : x \in A, x > x_0\}$$

(since F is strictly increasing and bounded on A). It follows that $F(x_0) = \inf\{F(x) : x \in A, x > x_0\}$, so $F(x_0) \in [F(a), F(b)]$ and $F(A \cup A_+) = [F(a), F(b)]$.

Let $a < x_1 < x_2 \leq b_0,\ x_2 \in A_+$. Since F is strictly right increasing$_*$ on A, $F(x_1) < F(x)$, for each $x \in A, x > x_2$, hence $F(x_1) \leq \inf\{F(x) : x \in A, x > x_2\} = F(x_2)$. Thus F is right increasing$_*$ on $A \cup A_+$.

Lemma 3 *Let $F : [a, b] \to \mathbb{R}, F(a) < F(b), F \in \mathcal{C}_d$ on $[a, b]$.*

(i) If $a_0 = \sup\{x \in [a, b] : F(x) = F(a)\}$ then $F(a_0) = F(a)$ and $F(x) > F(a_0)$, for each $x \in (a_0, b]$.

(ii) If $b_0 = \inf\{x \in [a_0, b] : F(x) = F(b)\}$, then $a_0 < b_0, F(b_0) = F(b)$ and $F(x) < F(b_0)$, for each $x \in (a_0, b_0]$.

(iii) $F([a_0, b_0]) = [F(a_0), F(b_0)]$.

(iv) There exists a set $A \subset [a_0, b_0]$ such that $a_0, b_0 \in A, F$ is strictly right increasing$_*$ on A and $F(A) = [F(a), F(b)]$.

(v) F is right increasing$_*$ on $A \cup A_+$ and $F(A \cup A_+) = [F(a), F(b)]$, where $A_+ = \{x : x$ is a right accumulation point of $A\}$.

(vi) There exists a set $B \subset [a_0, b_0]$, such that $a_0, b_0 \in B, F$ is strictly left increasing$_*$ on B, and $F(B) = [F(a), F(b)]$.

(vii) F is left increasing$_*$ on $B \cup B_-$ and $F(B \cup B_-) = [F(a), F(b)]$, where $B_- = \{x : x$ is a left accumulation point of $B\}$.

Lemma 4 *Let $F : [a, b] \to \mathbb{R}, F(a) < F(b)$, and let $P = \{x : F'(x) \geq 0\}$. If $F \in \mathcal{C}_d$ on $[a, b]$ and $|F(P)| = 0$ then there exist $E \subset [a, b]$ and $K \subset [F(a), F(b)]$, such that:*

(i) $|E| = 0$ and $|K| > (F(b) - F(a))/2$;

(ii) E and K are compact sets;

(iii) $F(E) = K$;

(iv) F *is strictly increasing on* E.

By Theorem 4, $F \in \mathcal{D}_+$ on $[a,b]$. Let $a_0 = \sup\{x \in [a,b] : F(x) = F(a)\}$ and $b_0 = \inf\{x \in [a,b] : F(x) = F(b)\}$. By Lemma 3, (i), (ii), (iii), $F(a_0) = F(a), F(b_0) = F(b)$ and $F([a_0,b_0]) = [F(a_0), F(b_0)]$. By Lemma 3, (iv), there exists a set $A \subset [a_0,b_0], a_0, b_0 \in A$, such that F is strictly right increasing$_*$ on A and

$$F(A) = [F(a_0), F(b_0)]. \tag{1}$$

We show that $\overline{A}$ is nowhere dense. Suppose on the contrary that there exists an interval $[c,d]$ with endpoints in A, such that $\overline{A} \supset [c,d]$. It follows that $[c,d) \subset A \cup A_+$. By Lemma 3, (v), F is increasing on $[c,d], F(c) < F(d)$ and $F([c,d]) = [F(c), F(d)]$. By Theorem 1, $|F(P)| = F(d) - F(c)$. This contradicts the fact that $|F(P)| = 0$. Hence $\overline{A}$ is nowhere dense. Let $\{(c_k, d_k)\}, k = 1,2,\ldots,$ be the intervals contiguous to $\overline{A}$. ($k = 1,2,\ldots,$, since $\overline{A}$ is nowhere dense). Then $A_0 \subset A \cup A_+$, where $A_0 = \overline{A} - (\bigcup_{k=1}^{\infty}\{c_k\})$, and F is right increasing$_*$ on A_0 (see Lemma 3, (v)). Let $A_1 = \{x \in A_0 : \underline{D}^+F(x) = -\infty\}$. We show that

$$|F(a_1)| = F(b) - F(a) = F(b_0) - F(a_0). \tag{2}$$

Since F is right increasing$_*$ on A_0, $\underline{D}^-F(x) \geq 0, x \in A_0$. Let $B_0 = A_0 - A_1$. Then $\underline{D}^+F(x) > -\infty$ on B_0. Hence $\underline{F}'(x) > -\infty$ on B_0. By Theorem 2, $F \in VB^*G$ on B_0. By Theorem 1, since $|F(P)| = 0$, it follows that $|F(B_0)| = 0$, Since $F(A) = [F(a), F(b)]$, we have (2). Let $e \in (0,1)$. Let N be a natural number such that

$$\sum_{k=N}^{\infty}(d_k - c_k) < \frac{(b_0 - a_0)}{2}. \tag{3}$$

We shall construct a cover in the Vitali sense for the set $F(A_1)$.
Let $x \in A_1$. Then $\underline{D}^+F(x) = -\infty$. It follows that there exist

$$k(x) \geq N \text{ and } \alpha(x) \in [c_{k(x)}, d_{k(x)}) \tag{4}$$

such that $F(\alpha(x)) < F(x)$. Since F is right increasing$_*$ on A_0,

$$F(x) \leq F(d_{k(x)}). \tag{5}$$

For each $k(x)$ choose $\epsilon(x)$ so that

$$0 < \epsilon(x) < \min\{F(x) - F(\alpha(x)) \,;\, \frac{(F(b) - F(a))}{4}\}.$$

Let $J_{x,\epsilon(x)} = [F(x) - \alpha(x), F(x)]$. Then

$$\{J_{x,\epsilon(x)}\}, x \in A_1, \epsilon(x) \in (0, \min\{F(x) - F(\alpha(x)); \frac{(F(b) - F(a))}{4}\}$$

is a cover in the Vitali sense for the set $F(A_1)$. Since $F(d_{k(x)}) \geq F(x) > F(x) - \epsilon(x) > F(\alpha(x))$ and $F \in \mathcal{D}_+$, it follows that the set $\{t \in [\alpha(x), d_{k(x)}] : F(t) = F(x) - \epsilon(x)\} \neq \emptyset$. Let $a_{x,\epsilon(x)} = \sup\{t \in [\alpha(x), d_{k(x)}] : F(t) = F(x) - \epsilon(x)\}$. By Lemma 3, (i), $F(a_{x,\epsilon(x)}) = F(x) - \epsilon(x)$. Since $F \in \mathcal{D}_+$, it follows that the set $\{t \in [a_{x,\epsilon(x)}, d_{k(x)}] : F(t) = F(x)\} \neq \emptyset$. Let $b_{x,\epsilon(x)} = \inf\{t \in [a_{x,\epsilon(x)}, d_{k(x)}] : F(t) = F(x)\}$. By Lemma 3, (ii), $F(b_{x,\epsilon(x)}) = F(x)$ and $a_{x,\epsilon(x)} < b_{x,\epsilon(x)}$. By Lemma 3, (iii), $F([a_{x,\epsilon(x)}, b_{x,\epsilon(x)}]) = [F(a_{x,\epsilon(x)}), F(b_{x,\epsilon(x)})] = J_{x,\epsilon(x)}$. By Theorem 3, there exist a natural number n ,$x_i \in A_1$, and $\epsilon(x_i) \in (0, \min\{F(x_i) - F(\alpha(x_i)); \frac{F(b)-F(a))}{4}\}, i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} |J_{x_i}, \epsilon(x_i)| > (F(b) - F(a)) \cdot (1 - e)$ and

$$\{J_{x_i}, \epsilon x_i\}, i = 1, 2, \ldots, n \text{ are pairwise disjoint} \tag{6}$$

We may suppose without loss of generality that

$$F(x_1) < F(x_2) < \ldots < F(x_n). \tag{7}$$

It follows that there exist a natural number $q \leq n$ and a set

$$\{r_1, r_2, \ldots, r_q\} \subset \{1, 2, \ldots, n\}, r_1 < r_2 < \ldots < r_q = n, r_0 = 0$$

such that

$$\begin{aligned} F(x_{r_0+1}) &\leq F(x_{r_1}) \leq F(d_{k(x_{r_0+1})}) < F(x_{r_1+1}) \leq F(x_{r_2}) && (8)\\ &\leq F(d_{k(x_{r_1+1})}) < \ldots < F(x_{r_{q-1}+1}) \leq F(x_{r_q}) && (9)\\ &= F(x_n) \leq F(d_{k(x_{r_{q-1}+1})}). && (10) \end{aligned}$$

Let $a_1 = a_{x_{r_0+1},\epsilon(x_{r_0+1})}$ and $b_1 = b_{x_{r_0+1},\epsilon(x_{r_0+1})}$. Then

$$[a_1, b_1] \subset [c_{k(x_{r_0+1})}, d_{k(x_{r_0+1})}]. \tag{11}$$

In general let $a_{r_1} = \sup\{t \in [b_{r_1-1}, d_{k(x_{r_0+1})}] : F(t) = F(x_{r_1}) - \epsilon(x_{r_1})\}$ and $b_{r_1} = \inf\{t \in [a_{r_1}, d_{k(x_{r_0+1})}] : F(t) = F(x_{r_1})\}$. By Lemma 3, (i), (ii), (iii),

$$\begin{aligned} F(a_i) &= F(x_i) - \epsilon(x_i), F(b_i) = F(x_i) \text{ and} && (12)\\ F([a_i, b_i]) &= [F(a_i), F(b_i)], i = 1, 2, \ldots, r_1. && (13) \end{aligned}$$

By (6), (10), (9),

$a_1 < b_1 < a_2 < b_2 < \ldots < a_{r_1} < b_{r_1}$ and $[a_i, b_i] \subset [c_{k(x_{r_0+1})}, d_{k(x_{r_0+1})}]$.

Let $a_{r_1+1} = a_{x_{r_1+1,\epsilon(x_{r_1+1})}}$ and $b_{r_1+1} = b_{x_{r_1+1,\epsilon(x_{r_1+1})}}$. Then

$$[a_{r_1+1}, b_{r_1+1}] \subset [c_{k(x_{r_1+1})}, d_{k(x_{r_1+1})}]. \tag{14}$$

Continuing, we obtain

$$\begin{aligned} a_1 &< b_1 < \ldots < a_{r_1} < b_{r_1} < a_{r_1+1} < b_{r_1+1} < \ldots && (15)\\ &< a_{r_q-1} < b_{r_q-1} < a_{r_q} = a_n < b_{r_q} = b_n; && (16)\end{aligned}$$

$$\begin{aligned} F(a_i) &= F(x_i) - \epsilon(x_i), F(b_i) = F(x_i) \text{ and} && (17)\\ F([a_i, b_i]) &= [F(a_i), F(b_i)], i = 1, 2, \ldots, n; && (18)\end{aligned}$$

$$[a_i, b_i] \subset [c_{k(x_j+1)}, d_{k(x_{r_j+1})}], i = r_j + 1, \ldots, r_{j+1}; j = 0, \ldots, q-1. \tag{19}$$

By (3), (4), (15), we have

$$\sum_{i=1}^{n} (b_i - a_i) < \frac{(b_0 - a_0)}{2}. \tag{20}$$

Moreover, we have

$$\sum_{i=1}^{n} (F(b_i) - F(a_i)) > (F(b_0) - F(a_0)) \cdot (1 - e) \text{ and} \tag{21}$$

$$F(b_i) - F(a_i) < \frac{(F(b_0) - F(a_0))}{4}, i = 1, 2, \ldots, n. \tag{22}$$

Let $e_i \in (0, 1), i = 1, 2, \ldots$, such that

$$(1 - e_1) \cdot (1 - e_2) \cdot (1 - e_3) \cdot \ldots > \frac{1}{2}. \tag{23}$$

Suppose we have constructed the sets E_{p-1} and $K_{p-1}, p \geq 2$.

$$E_{p-1} = (i_1, \ddot{\cup}, i_{p-1})[a_{i_1} \ldots i_{p-1}, b_{i_1} \ldots i_{p-1}],$$

$$K_{p-1} = (i_1, \ddot{\cup}, i_{p-1})[F(a_{i_1} \ldots i_{p-1}), F(b_{i_1} \ldots i_{p-1})],$$

with $|E_{p-1}| < (b_0 - a_0)/2^{p-1}$ and

$$|K_{p-1}| > (F(b_0) - F(a_0))(1 - e_1) \ldots (1 - e_{p-1}).$$

Similar to the construction of $E_1 \subset [a_0, b_0]$ and K_1, for $e_p \in (0,1)$ and $[a_{i_1} \dots i_{p-1}, b_{i_1} \dots i_{p-1}]$, there exist natural number $n_{i_1}, \dots, i_{p-1}$ and some pairwise disjoint subintervals $[a_{i_1} \dots i_{p-1}, b_{i_1} \dots i_{p-1}], i_p = i, \dots, n_{i_1}, \dots, i_{p-1}$ (numbered from the left to the right), such that the sum of their length is less than $(b_{i_1} \dots i_{p-1} - a_{i_1} \dots i_{p-1})/2$, and for $i_p = 1, 2, \dots, n_{i_1} \dots i_{p-1}$,

$$F([a_{i_1} \dots i_p, b_{i_1} \dots i_p]) = [F(a_{i_1} \dots i_p), F(b_{i_1} \dots i_p)]$$

are pairwise disjoint closed subintervals of $[F(a_{i_1} \dots i_p), F(b_{i_1} \dots i_p)]$. These intervals are in an increasing order on the y-line and each of them has the length less that $(F(b_0) - F(a_0))/4^p$. The sum of their length is less than $(F(b_{i_1} \dots i_{p-1}) - F(a_{i_1} \dots i_{p-1}))(1 - e_p)$. Let

$$E_p = \bigcup_{(i_1, \dots, i_{p-1})} [a_{i_1} \dots i_p, b_{i_1} \dots i_p]$$

and

$$K_p = \bigcup_{(i_f 1, \dots, i_{p-1})} [F(a_{i_1} \dots i_p), F(b_{i_1} \dots i_p)].$$

Then $|E_p| < (b_0 - a_0)/2^p$ and $|K_p| > (F(b_0) - F(a_0))(1 - e_1) \dots (1 - e_p)$. Let $E = \cup_{p=1}^{\infty} E_p$ and $K = \cup_{p=1}^{\infty} K_p$.

(i) It follows that $|E| = 0, K_p$ is measurable and $|K| > F(b_0) - F(a_0))/2 = (F(b) - F(a))/2$.

(ii) Since $E_p, K_p, p \geq 1$, are compact sets, it follows that E and K are also compact sets.

(iii) Let $x_0 \in E$. Then there exists a sequence of closed intervals $[a_{i_1 \dots i_p}, b_{i_1 \dots i_p}], p \geq 1$, each interval containing x_0. Then we have $x_0 = \bigcap_{\substack{i_1, i_2, \dots, i_p \\ p \geq 1}} [a_{i_1 i_2 \dots i_p}, b_{i_1 i_2 \dots i_p}]$ and

$$\begin{aligned} F(x_0) &\in \bigcap_{\substack{i_1, \dots, i_p \\ p \geq 1}} F([a_{i_1 \dots i_p}, b_{i_1 \dots i_p}]) \\ &= \bigcap_{\substack{i_1, \dots, i_p \\ p \geq 1}} [F(a_{i_1 \dots i_p}), F(b_{i_1 \dots i_p})] \subset \bigcap_{p \geq 1} K_p = K \end{aligned}$$

hence $F(E) \subset K$.

To see that $K \subset F(E)$, let $y_0 \in K$. Then there exists a sequence of closed intervals $\{[a_{i_1 \ldots i_p}, b_{i_1 \ldots i_p}]\}$ with $p \geq 1$, such that each interval $[F(a_{i_1 \ldots i_p}), F(b_{i_1 \ldots b i_p})]$ contains y_0. Then

$$y_0 = \bigcap_{\substack{i_1, \ldots, i_p \\ p \geq 1}} [F(a_{i_1 \ldots i_p}), F(b_{i_1 \ldots b i_p})].$$

It follows that

$$\bigcap_{\substack{i_1, \ldots, i_p \\ p \geq 1}} [a_{i_1 \ldots i_p}, b_{i_1 \ldots b i_p}]$$

degenerates to a point x_0. Hence,

$$F(x_0) \in \bigcap_{\substack{i_1, \ldots, i_p \\ p \geq 1}} [F(a_{i_1 \ldots i_p}), F(b_{i_1 \ldots b i_p})] = \{y_0\}$$

and $F(x_0) = y_0$. It follows that $K \subset F(E)$

(iv) Let $x' < x'', x', x'' \in E$. Then there exist two sequences of closed intervals $\{[a_{i'_1 \ldots i'_p}, b_{i'_1 \ldots i'_p}]\}, p \geq 1$ and $\{[a_{i''_1 \ldots i''_p}, b_{i''_1 \ldots i''_p}]\}, p \geq 1$, such that both

$$x' = \bigcap_{\substack{i'_1, \ldots, i'_p \\ p \geq 1}} [a_{i'_1 \ldots i'_p}), b_{i'_1 \ldots b i'_p})]$$

and

$$x'' = \bigcap_{\substack{i''_1, \ldots, i''_p \\ p \geq 1}} [a_{i''_1 \ldots i''_p}, b_{i''_1 \ldots b i''_p}]$$

Since $x' < x''$, there exists $p_0 \geq 1$, such that $i'_j = i''_j, j = 1, 2, \ldots, p_0 - 1$ and $i'_{p_0} < i''_{p_0}$. Hence

$$x' \in [a_{i'_1 \ldots i'_{p_0}}, b_{i'_1 \ldots b i'_{p_0}}] \text{ and } x'' \in [a_{i''_1 \ldots i''_{p_0 - 1} i''_{p_0}}, b_{i''_1 \ldots i''_{p_0 - 1} i''_{p_0}}].$$

It follows that

$$F(a_{i'_1 \ldots i'_{p_0 - 1} i'_{p_0}}) < F(x') < F(b_{i'_1 \ldots i'_{p_0 - 1} i'_{p_0}})$$

$$F(a_{i'_1 \ldots i'_{p_0 - 1} i''_{p_0 - 1}}) < F(x'') < F(b_{i'_1 \ldots i'_{p_0 - 1} i''_{p_0}}),$$

hence $F(x') < F(x'')$.

Definition 4 *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b]$. F is said to $\overline{AC}$ on P, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $\sum(F(b_i) - F(a_i)) < \epsilon$, for each sequence of nonoverlapping closed intervals $\{[a_i, b_i]\}, i \geq 1$, with endpoints in P and $\sum(b_i - a_i) < \delta$. Let $\overline{AC} = \{F : -F \in \overline{AC}\}$. Clearly $AC = \underline{AC} \cap \overline{AC}$ on P, where AC is the class of all absolutely continuous functions on P.*

Definition 5 *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b]$. F is said to satisfy Foran's condition (M) on P, if $F \in AC$ on E, whenever $E \subset P$ and $F \in VB$ on P.*

Definition 6 *Let $F : [a,b] \to \mathbb{R}, P \subset [a,b]$. F is said to be $(\underline{M})$ on P, if $F \in \underline{AC}$ on E, whenever $E \subset P$ and $F \in VB$ on E. Let $(\underline{M}) = \{F : -F \in (\underline{M})\}$. Clearly $(M) = (M) \cap (\overline{M})$ on P.*

Theorem 5 (An extension of Theorem 9 of [1].) *Let $F : [a,b] \to \mathbb{R}$, and $F \in C_d \cap (\overline{M})$ on $[a,b]$. Then F is derivable on a set of positive measure. Moreover, if there exist $0 \leq a < b \leq 1$, such that $F(a) < F(b)$, then $|F(P)| > 0$, where $P = \{x : F'(x) \geq 0\}$.*

The proof is similar to that of Theorem 9 of [1], using Lemma 4 instead of Lemma 6 of [1].

Theorem 6 (An extension of Theorem 10 of [1].) *Let $F : [a,b] \to \mathbb{R}$, and $F \in C_d \cap (\overline{M})$ on $[a,b]$. If $F'(x) \leq 0$ at almost every point x where $F'(x)$ exists and is finite, then F is decreasing on $[a,b]$.*

The proof is similar to that of Theorem 10 of [1].

Corollary 1 (An extension of Corollary 3 of [1].) *Let $F : [a,b] \to \mathbb{R}$, and $F \in C_d \cap (M)$ on $[a,b]$. If $F'(x) \geq 0$ at almost every point x where $F'(x)$ exists and is finite, then F is AC and increasing on $[a,b]$.*

We are indebted to the reviewer for many remarks allowing to improve the final version of the text.

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