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ON CONTINUOUS AND QUASI-CONTINUOUS FUNCTIONS

It is not hard to see (cf. e.g. [4]) that, in the case of real functions of a real variable, every discontinuous Darboux function has infinite variation. On the other hand,it is possible for a Darboux function $f : \mathbb{R}^2 \to \mathbb{R}^2$ to have finite variation, and be discontinuous at any point of its domain, as the following example shows.

Example 1 Let $g : \mathbb{R} \to [0,1]$ be a function mapping every non-degenerate interval to $[0,1]$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f((x,y)) = (g(x), 0)$.

However, rather surprising is (in view of the result of T. Šalat) R. Pawlak's result (cf. [3], Theorem 7) showing that, in the space of bounded Darboux functions $f : [0,1]^2 \to \mathbb{R}^2$ with finite variation, continuous functions constitute a boundary set. This result prompts one to if continuous functions constitute a 'small set' in the space of bounded Darboux functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ with finite variation? The question thus raised requires that the notion of a 'small set' be stated precisely. (Theorem 1.6 included in book [2] shows that a 'small set' in the sense of the Lebesgue measure may be 'large' in the topological sense, and a 'small set' in the topological sense may have a large Lebesgue measure).

It seems that, in the case of metric spaces, one may take porous sets as 'small sets'. Here it is essential to observe (cf. e.g. [5] as well as [6], Theorem 2.8) that if $A \subset \mathbb{R}$ is a porous set, then it is a nowhere dense set of Lebesgue measure zero and, thereby, the class of σ-porous sets is of the first category and is contained in the σ-ideal of sets of measure zero. In connection with these assertions, the question posed at the beginning may be reformulated as follows:

Do continuous functions constitute a porous set in the space of bounded Darboux functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ with finite variation, equipped with the metric of uniform convergence?

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The present paper includes the answer to the question posed here. What is more, it has been shown that continuous functions constitute a porous set in some space which, when considered as a subset of the space of bounded Darboux functions with finite variation, is a porous set.

Definition 1 *Let $f : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$, $\mathcal{Y}$ are arbitrary topological spaces. We say that f is a Darboux function if the image of each arc $\mathcal{L} \subset X$ is a connected set.*

Definition 2 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$. A function $\mathcal{N}_f : \mathbb{R}^2 \to \langle 0, +\infty\rangle$ defined in the following way: $\mathcal{N}_f(p)$ is equal to the number of elements of the set $f^{-1}(p)$ when the last set is finite or $+\infty$ when $f^{-1}(p)$ is not finite, is called the Banach indicatrix of the function f.*

Definition 3 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a Darboux function with measurable Banach indicatrix. We say that f has finite variation (in the Banach sense) if $\int_{\mathbb{R}^2} \mathcal{N}_f(p)\, dp < +\infty$. In this case set $V(f) = \int_{\mathbb{R}^2} \mathcal{N}_f(p)\, dp$.*

Throughout the paper, we adopt the standard symbols, definitions and notations. The letters $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote the sets of all positive integers, rational numbers and real numbers, respectively. The distance in the space $\mathbb{R}^2$ is denoted by d, whereas the letter ρ is used to write down the metric of uniform convergence in the space of bounded Darboux functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ with finite variation. The two-dimensional Lebesgue measure of the set $A \subset \mathbb{R}^2$ is denoted by $m_2(A)$.

Theorem 1 *In the space of bounded Darboux functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ having finite variation, endowed with the metric of uniform convergence, the set of functions possessing a dense set of points of quasi-continuity is a porous set.*

PROOF. Let f_0 be a bounded Darboux function with finite variation, possessing a dense set of points of quasi-continuity. Let $\varepsilon \in (0, 1)$, let x_0 be any point of quasi-continuity of the function f_0 and set $\alpha_0 = f_0(x_0)$. Then there exist a point $z \in \mathbb{R}^2$ and a number $\delta > 0$, such that

$$f_0(\bar{K}(z,\delta)) \subset K\big(\alpha_0, \frac{\varepsilon}{3}\big) \tag{1}$$

the ball with center β and radius $\frac{\varepsilon}{3}$. Let β be a point of the sphere $S\big(\alpha_0, \frac{\varepsilon}{3}\big)$, such that

$$\begin{array}{l} f_0(z) \in [\alpha_0, \beta) \text{ if } f_0(z) \neq \alpha_0; \\ \beta \text{ is any point of the sphere } S\big(\alpha_0, \frac{\varepsilon}{3}\big) \text{ if } f_0(z) = \alpha_0. \end{array} \tag{2}$$

Let $\mathcal{M} = \cup_{q\in(0,\frac{\varepsilon}{3})\cap\mathbb{Q}} S(\alpha_0, q) \cup [\alpha_0, \beta)$. Clearly $\mathcal{M}$ is a connected set, and $m_2(\mathcal{M}) = 0$. Let Θ be the family of all spheres $S(z,\eta)$ where $\eta \in (0,\delta)$. Define on Θ the equivalence relation '$\sim$' by $S(z,\eta_1) \sim S(z,\eta_2) \Leftrightarrow \eta_1 - \eta_2 \in \mathbb{Q}$. Let $\mathcal{P}$ be the set of equivalence classes. Then the cardinality of $\mathcal{P}$ is equal to the cardinality of the continuum.

Note that

(3) for any η_1, η_2 with $0 < \eta_1 < \eta_2 < \delta$, the family $\{S(z,\eta)\}_{\eta\in[\eta_1,\eta_2]}$ contains representatives of all equivalence classes (More precisely for any $w \in \mathcal{P}$, there exists $\eta \in [\eta_1, \eta_2]$ such that $S(z,\eta)$ belongs to w).

Let $t : \mathcal{P} \to \mathcal{M}$ be a bijection. Define $g_0 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g_0(x) = \begin{cases} t([S(z, d(z,x))]_\sim) & \text{if } x \in K(z,\delta)\backslash\{z\} \\ f_0(x) & \text{if } x \notin K(z,\delta)\backslash\{z\}, \end{cases}$$

where $[S(z,\eta])_\sim$ denotes the equivalence class containing $S(z,\eta)$. We shall first show that g_0 is a Darboux function. So assume that $\mathcal{L}$ is any fixed arc contained in $K(z,\delta)$. We shall show that

(4)
a) if there is a real number $r > 0$ such that $\mathcal{L} \subset (z,r)$, then $g_0(\mathcal{L})$ is a one-point set;
b) if there is no real number $r > 0$ such that $\mathcal{L} \subset S(z,r)$, then $g_0(\mathcal{L}) = \mathcal{M}$.

Obviously condition (4) implies the connectedness of the set $g_0(\mathcal{L})$.

To prove (4), let us first assume that $z \notin \mathcal{L}$. Let $r_1 = \inf_{a\in\mathcal{L}} d(z,a)$ and $r_2 = \sup_{a\in\mathcal{L}} d(z,a) < \delta$. Of course, for each $a \in \mathcal{L}$, we have $d(z,a) \in [r_1, r_2]$ and $r_1 \leq r_2$. If $r_1 = r_2$, then $\mathcal{L} \subset S(z,r_1)$ and, in view of the definition of the mapping g_0, the set $g_0(\mathcal{L})$ is one-element. So assume that $r_1 < r_2$. Since $\mathcal{L}$ is compact and connected and since the distance of a point from a set is a continuous function, we infer that

(5) for any $r \in [r_1, r_2]$, there exists an element $a_r \in \mathcal{L}$ such that $d(z,a_r) = r$.

By the definition of g_0, (5) and (3) imply

$$\begin{aligned} g_0(\mathcal{L}) &= \cup_{r_1\leq r\leq r_2} \cup_{a\in\mathcal{L}\cap S(z,r)} \{g_0(a)\} = \cup_{r_1\leq r\leq r_2}\{g_0(a_r)\} \\ &= \cup_{r_1\leq r\leq r_2}\{t([S(z,r)]_\sim)\} = \cup_{r\in(0,\delta)}\{t([S(z,r)]_\sim)\} = \mathcal{M}. \end{aligned}$$

The proof of (4) has thus been concluded in the case when $z \notin \mathcal{L}$.

So now assume that $z \in \mathcal{L}$. (We still have the assumption $\mathcal{L} \subset K(z,\delta)$.) Then there exists an arc $\tilde{\mathcal{L}} \subset \mathcal{L}\backslash\{z\}$ contained in no sphere $S(z,\eta)$ $(0 < \eta < \delta)$. By the proof of condition (4) carried out before, we may deduce that

$$g_0(\mathcal{L}) \supset g_0(\tilde{\mathcal{L}}) = \mathcal{M}. \tag{6}$$

Then by (2) we have $g_0(\mathcal{L}) \subset \{f_0(z)\} \cup \mathcal{M} = \mathcal{M}$. The above inclusion and (6) prove that $g_0(\mathcal{L}) = \mathcal{M}$ completing the proof of (4).

Next assume that $\mathcal{L}$ is an arc having points in common with the complement of $K(z,\delta)$. If $\mathcal{L} \cap K(z,\delta) = \emptyset$, then from the definition of the function g_0 it follows that $g_0(\mathcal{L}) = f_0(\mathcal{L})$ is connected. Thus in order to finish the proof of the Darboux property of g_0, it suffices to consider the case

$$\mathcal{L} \cap K(z,\delta) \neq \emptyset \qquad \text{and} \qquad \mathcal{L}\backslash K(z,\delta) \neq \emptyset. \tag{7}$$

First show that $g_0(\mathcal{L} \cap \bar{K}(z,\delta))$ is connected. To that end note that

(8) there exists an arc $\mathcal{L}' \subset \mathcal{L} \cap K(z,\delta)$ such that $\mathcal{L}'\backslash S(z,r) \neq \emptyset$ for any $r \in (0,\delta)$.

By (7) there exists $a \in \mathcal{L} \cap K(z,\delta)$. Thus there exists $\delta^* > 0$ such that $K(a,\delta^*) \subset K(z,\delta)$. Hence

$$a \in K(z, \delta - \frac{\delta^*}{2}) \cap \mathcal{L}. \tag{9}$$

The set $\bar{K}(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L}$ is a nonempty, closed subset of the Hausdorff continuum $\mathcal{L}$. Consequently the component C_a of this set containing a satisfies $C_a \cap \mathrm{Fr}_{\mathcal{L}}\left(\bar{K}(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L}\right) \neq \emptyset$. Hence it appears that

$$C_a \cap S(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L} \neq \emptyset. \tag{10}$$

Indeed, to prove (10), it suffices to demonstrate that

$$S(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L} \supset \mathrm{Fr}_{\mathcal{L}}\left(\bar{K}(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L}\right). \tag{11}$$

So, let $x \in \mathrm{Fr}_{\mathcal{L}}\left(\bar{K}(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L}\right)$. Then

$$x \in \bar{K}(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L} \tag{12}$$

and

$$\mathrm{Int}_{\mathcal{L}}\left(\bar{K}(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L}\right) \supset K(z,\delta - \frac{\delta^*}{2}) \cap \mathcal{L}. \tag{13}$$

If $x \notin S(z, \delta - \frac{\delta^*}{2}) \cap \mathcal{L}$, then by (12) $x \notin S(z, \delta - \frac{\delta^*}{2})$ and consequently again by (12) $x \in K(z, \delta - \frac{\delta^*}{2}) \cap \mathcal{L}$. Then (13) proves that $x \notin \mathrm{Fr}\,(\bar{K}(z, \delta - \frac{\delta^*}{2}) \cap \mathcal{L})$, which is contrary to the definition of x. Thus $x \in S(z, \delta - \frac{\delta^*}{2})$, verifying (11) and, thereby, (10). By (9) and (10), C_a is not a one-element set and $C_a \backslash S(z, r) \neq \emptyset$ for $r \in (0, \delta)$. Let $\mathcal{L}' = C_a$. Then $\mathcal{L}'$ is a closed, connected subset of the arc $\mathcal{L}$ consisting of more than one element. Thus $\mathcal{L}'$ is also an arc. Therefore (8) is proved.

In view of (8) and (4) we may deduce that

$$g_0(\mathcal{L} \cap K(z, \delta)) = \mathcal{M}. \tag{14}$$

The equality $g_0\big|_{S(z,\delta)} = f_0\big|_{S(z,\delta)}$ as well as (1) and (14) imply, in turn, that

$$g_0(\bar{K}(z, \delta) \cap \mathcal{L}) \subset K(\alpha_0, \frac{\varepsilon}{3}). \tag{15}$$

Relations (14) and (15) allow us to conclude that $\mathcal{M} \subset g_0(\bar{K}(z, \delta) \cap \mathcal{L}) \subset \tilde{\mathcal{M}}$. On the basis of the above inclusions, the connectedness of $\mathcal{M}$ implies that

$$g_0(\bar{K}(z, \delta) \cap \mathcal{L}) \text{ is a connected set.} \tag{16}$$

We now prove that $g_0(\mathcal{L})$ is a connected set. Let $\{C_s\}_{s \in S}$ denote the family of all components of the set $\mathcal{L} \backslash K(z, \delta)$ and let s be any fixed element of the set S. It is easy to see that C_s is an arc or a one-element set, and since (cf. (7)) $\mathcal{L} \cap S(z, \delta) \neq \emptyset$,

$$C_s \cap S(z, \delta) \neq \emptyset. \tag{17}$$

Indeed, note that $\mathcal{H} = \mathcal{L} \backslash K(z, \delta)$ is a non-empty (cf. (7)), closed set in $\mathcal{L}$. In turn $\mathcal{L} \backslash \bar{K}(z, \delta)$ is a subset of $\mathcal{H}$ open in $\mathcal{L}$. Thus $\mathrm{Int}_{\mathcal{L}} \mathcal{H} \supset \mathcal{L} \backslash \bar{K}(z, \delta)$. Since $\mathcal{H}$ is closed in $\mathcal{L}$, we infer $\mathrm{Fr}_{\mathcal{L}} \mathcal{H} \subset \mathcal{L} \cap S(z, \delta)$. So $C_s \cap \mathrm{Fr}_{\mathcal{L}} \mathcal{H} \neq \emptyset$ implies (17). Moreover

$$g_0(C_s) = f_0(C_s) \text{ is a connected set} \tag{18}$$

and (by (17))

$$g_0(C_s) \cap g_0(\mathcal{L} \cap \bar{K}(z, \delta)) \neq \emptyset. \tag{19}$$

By (16), (18) and (19) and since $s \in S$ is arbitrary,

$$g_0(C_s) \cup g_0(\mathcal{L} \cap \bar{K}(z, \delta)) \text{ is a connected set for each } s \in S. \tag{20}$$

Since $g_0(\mathcal{L}) = g_0(\mathcal{L} \cap \bar{K}(z, \delta)) \cup \bigcup_{s \in S} g_0(C_s)$, it follows that $g_0(\mathcal{L})$ is connected and therefore the proof that g_0 is a Darboux function is complete.

Next we define a new mapping $k_0 : \mathbb{R}^2 \to \mathbb{R}^2$. To that end let Φ be the family of all spheres $S(\alpha_0, r)$ where $r \in (0, \frac{\varepsilon}{3}) \setminus \mathbb{Q}$. Define an equivalence relation '$\approx$'by

$$\text{for any } S(\alpha_0, r_1), S(\alpha_0, r_2) \in \Phi S(\alpha_0, r_1) \approx S(\alpha_0, r_2)$$
$$\text{if and only if } r_1 - r_2 \text{ is a rational number.}$$

This relation divides Φ into a continuum of equivalence classes. Denote the set of these classes by Φ^*. Then there exists a surjection $l : \Phi^* \to \mathcal{M}$. Define $k_0 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$k_0(x) = \begin{cases} l([S(\alpha_0, d(\alpha_0, x))]_\approx) & \text{when there is } \varphi \in \Phi \text{ such that } x \in \varphi \\ x & \text{when } x \notin \varphi \text{ for any } \varphi \in \Phi, \end{cases}$$

where $[S(\alpha, r)]_\approx$ denotes the class containing $S(\alpha, r)$. It is not hard to show that $k_0(K(\alpha_0, \frac{\varepsilon}{3})) = \mathcal{M}$. We shall now establish that

(21) $\quad k_0(C)$ is a connected set for any connected $C \subset \mathbb{R}^2$.

Let $C \subset \mathbb{R}^2$ be a connected set. If $C \subset \mathbb{R}^2 \backslash K(\alpha_0, \frac{\varepsilon}{3})$, then $k_0(C) = C$ and condition (21) is satisfied. Consequently assume now that $C \subset K(\alpha_0, \frac{\varepsilon}{3})$. Then the following situations are possible:

1^0 There exists $r \in (0, \frac{\varepsilon}{3})$ such that $C \subset S(\alpha_0, r)$. If r is an irrational number, then $k_0(C)$ is a one-element set and thus a connected set. If r is a rational number, then $k_0(C) = C$ again a connected set.

2^0 $C = \{\alpha_0\}$. Then $k_0(C) = \{\alpha_0\}$ is, of course, a connected set.

3^0 There are distinct $r_1, r_2 \in (0, \frac{\varepsilon}{3})$ with $C \cap S(\alpha_0, r_1) \neq \emptyset \neq C \cap S(\alpha_0, r_2)$. In this case it is easy to verify that C contains elements of the representatives of all equivalence classes. Thus the definition of k_0 implies that $k_0(C) = \mathcal{M}$, which means that once again $k_0(C)$ is a connected set.

Finally assume $C \cap K(\alpha_0, \frac{\varepsilon}{3}) \neq \emptyset$ and $C \backslash K(\alpha_0, \frac{\varepsilon}{3}) \neq \emptyset$. Then $\mathcal{M} \subset \mathcal{M} \cup (C \cap S(\alpha_0, \frac{\varepsilon}{3})) \subset \bar{\mathcal{M}} = \bar{K}(\alpha_0, \frac{\varepsilon}{3})$. From the connectedness of $\mathcal{M}$ it follows that

(22) $\quad \mathcal{M} \cup (C \cap S(\alpha_0, \dfrac{\varepsilon}{3}))$ is a connected subset of $\mathcal{M} \cup C$.

In addition $(\mathcal{M} \cup C) \backslash (\mathcal{M} \cup (C \cap S(\alpha_0, \frac{\varepsilon}{3}))) = \mathcal{F} \cup \mathcal{G}$ where $\mathcal{F} = C \backslash \bar{K}(\alpha_0, \frac{\varepsilon}{3})$ and $\mathcal{G} = (C \cap K(\alpha_0, \frac{\varepsilon}{3})) \backslash \mathcal{M}$. Moreover $\mathcal{F} \subset \mathbb{R}^2 \backslash \bar{K}(\alpha_0, \frac{\varepsilon}{3})$ and $\mathcal{G} \subset K(\alpha_0, \frac{\varepsilon}{3})$. Hence $\mathcal{F}$ and $\mathcal{G}$ are separated sets (also in $\mathcal{M} \cup C$). In view of the equality $\mathcal{M} \cup (C \cup S(\alpha_0, \frac{\varepsilon}{3})) \cup \mathcal{F} = \mathcal{M} \cup (C \backslash K(\alpha_0, \frac{\varepsilon}{3}))$ and by Theorem 8, (page 228, [1]), we deduce that

(23) $\quad \mathcal{M} \cup (C \backslash K(\alpha_0, \dfrac{\varepsilon}{3}))$ is a connected set in $\mathcal{M} \cup C$, and thus in $\mathbb{R}^2$.

The definition of k_0 implies the equalities

$$\begin{aligned} k_0(C) &= k_0(C \cup K(\alpha_0, \frac{\varepsilon}{3})) \cup k_0(C \backslash K(\alpha_0, \frac{\varepsilon}{3})) \\ &= \mathcal{M} \cup (C \backslash K(\alpha_0, \frac{\varepsilon}{3})). \end{aligned}$$

Hence and from relation (23) it follows that $k_0(C)$ is a connected set completing the proof of proposition (21).

By (21) and since g_0 has the Darboux property, $g_0' = k_0 \circ g_0$ is a Darboux function. It is not difficult to check that

$$\mathcal{N}_{g_0'}(y) = \begin{cases} \mathcal{N}_{f_0}(y) & \text{if } y \notin K(\alpha_0, \frac{\varepsilon}{3}), \\ +\infty & \text{if } y \in \mathcal{M}, \\ 0 & \text{if } y \in K(\alpha_0, \frac{\varepsilon}{3}) \backslash \mathcal{M}, \end{cases}$$

that $\mathcal{N}_{g_0'}$ is measureable and $\int_{\mathbb{R}^2} \mathcal{N}_{g_0'}(y)dy < +\infty$. Consequently g_0' has finite variation.

Since $\varepsilon \in (0,1)$, we have $\frac{\varepsilon}{3} - \frac{\varepsilon^2}{3} > 0$. We shall now show that

(24) no function possessing a dense set of points of quasi-continuity belongs to the ball $K(g_0', \frac{\varepsilon}{3} - \frac{\varepsilon^2}{3})$.

Let h be a function having a dense set of points of quasi-continuity. Then there exists a point p_0 of quasi-continuity of h, such that $p_0 \in K(z,\delta) \backslash \{z\}$. Let $\gamma_0 = h(p_0)$. Then there exist q_0 and $\delta_1 > 0$ such that

(25) $\delta_1 < d(z, q_0) < \delta,\ K(q_0, \delta_1) \subset K(z, \delta)$ and $h(K(q_0, \delta_1)) \subset K(\gamma_0, \frac{\varepsilon^2}{6})$.

Let

$$r_0 \in (\frac{\varepsilon}{3} - \frac{\varepsilon^2}{6}, \frac{\varepsilon}{3}) \cap \mathbb{Q}. \tag{26}$$

If $\alpha_0 \neq \gamma_0$, then let

$$\gamma_1 \in \hat{H}_{\alpha_0}^{\gamma_0} \cap S(\alpha_0, r_0) \tag{27}$$

where $\hat{H}_{\alpha_0}^{\gamma_0}$ denotes the half-line with the initial point α_0, not containing γ_0 and contained in the line passing through α_0 and γ_0. If $\alpha_0 = \gamma_0$, then assume

$$\gamma_1 \in S(\alpha_0, r_0). \tag{28}$$

Since $r_0 \in \mathbb{Q}$, by (27) and (28) $\gamma_1 \in \mathcal{M}$. Consequently from the definitions of k_0 and g_0

(29) there exists $z_1 \in K(z, \delta) \backslash \{z\}$ such that $\gamma_1 = g_0'(z_1)$.

Consider an equivalence class $[S(z,d(z,z_1))]_\sim$. Let $d(z,q_0)=a$. Then $S(z,r)\cap K(q_0,\delta_1)\neq\emptyset$ for any $r\in\left[a-\frac{\delta_1}{2},\,a+\frac{\delta_1}{2}\right]$. From (3) there exists $r_1\in\left[a-\frac{\delta_1}{2},\,a+\frac{\delta_1}{2}\right]$ such that $[S(z,r_1)]_\sim=[S(z,d(z,z_1))]_\sim$. Let $\hat{z}_1\in S(z,r_1)\cap K(q_0,\delta_1)$. Then from the definitions of g_0 and k_0 and since $r_0\in\mathbb{Q}$,

$$g_0'(\hat{z}_1)=g_0'(z_1)=\gamma_1. \tag{30}$$

Of course $\hat{z}_1\in K(q_0,\delta_1)$. Therefore (25) implies that $h(\hat{z}_1)\in K(\gamma_0,\frac{\varepsilon^2}{6})$. Hence from (26), (30) and since $\alpha_0\in[\gamma_0,\gamma_1]$,

$$\begin{aligned} d(g_0'(\hat{z}_1)h(\hat{z}_1)) &= d(\gamma_1,h(\hat{z}_1))\geq d(\gamma_1,K(\gamma_0,\frac{\varepsilon^2}{6}))\\ &= r_0+d(\alpha_0,\gamma_0)-\frac{\varepsilon^2}{6}>\frac{\varepsilon}{3}-\frac{\varepsilon^2}{3}+d(\alpha_0,\gamma_0)\geq\frac{\varepsilon}{3}-\frac{\varepsilon^2}{3}. \end{aligned}$$

Hence $\rho(g_0',h)\geq\frac{\varepsilon}{3}-\frac{\varepsilon^2}{3}$. Since the choice of h was arbitrary, (24) follows.

Finally by (1) and the definition of g_0', we conclude that $\rho(f_0,g_0')\leq\frac{2}{3}\varepsilon$. Thus $K(g_0',\frac{\varepsilon}{3}-\frac{\varepsilon^2}{3})\subset K(f_0,\varepsilon)$. Hence in the space of bounded Darboux functions with finite variation, the porosity of the set of all functions having a dense set of points of quasi-continuity is no smaller than $2\limsup_{\varepsilon\to0^+}\frac{\frac{\varepsilon}{3}-\frac{\varepsilon^2}{3}}{\varepsilon}=\frac{2}{3}>0$. $\square$

Remark As was suggested by the referee, Theorem 1 holds if we replace the density of the sets of points of quasicontinuity by that of the set of continuity points. It is so because each function $f:\mathbb{R}^2\to\mathbb{R}^2$ possessing a dense set of points of quasicontinuity also has a dense set of continuity points.

The above theorem answers the question raised in the introduction because it implies that the set of continuous functions in the space of bounded Darboux functions with finite variation constitutes a porous set. The analysis of various examples leads, however, to the question, "What can be said about the set of continuous functions in the space of bounded and quasi-continuous Darboux functions with finite variation?" In particular the earlier results suggest that this set may be porous. The theorem presented below shows that this conjecture is true.

Theorem 2 *In the space of quasi-continuous bounded Darboux functions $f:\mathbb{R}^2\to\mathbb{R}^2$ with finite variation, with the metric of uniform convergence, continuous functions constitute a porous set.*

PROOF. Let f_0 be any continuous function with finite variation. The proof will be carried out in two stages.

First assume that f_0 is a constant function, and that $\varepsilon\in(0,1)$. Let $f_0(\mathbb{R}^2)=\{\alpha_0\}=\{(\alpha_1^0,\alpha_2^0)\}$ and let $\beta\in S(\alpha_0,\frac{\varepsilon}{3})\cap\{(\xi,\eta):\eta=\alpha_2^0 \text{ and } \xi>$

$\alpha_1^0\} \subset \mathbb{R}^2$. Set $\varepsilon_1 = \frac{\varepsilon}{2} - \frac{\varepsilon^2}{4} > 0$ and $\varepsilon_2 = \frac{\varepsilon}{4}\sqrt{\varepsilon(4-\varepsilon)}$. Then $\varepsilon_1^2 + \varepsilon_2^2 = (\frac{\varepsilon}{2})^2$. Define $\varphi : (0, \varepsilon_2] \to \mathbb{R}$ be a function by $\varphi(\xi) = \varepsilon_1 \sin \frac{\pi\varepsilon_2}{\xi}$. Fix $x_0 \in \mathbb{R}^2$ and define $g_0 : \mathbb{R}^2 \to \mathbb{R}^2$by

$$g_0(x) = \begin{cases} \alpha_0 & \text{if } x = x_0 \\ (\alpha_1^0 + d(x, x_0), \alpha_2^0 + \varphi(d(x, x_0))) & \text{if } 0 < d(x, x_0) \le \varepsilon_2 \\ (\alpha_1^0 + d(x, x_0), \alpha_2^0) & \text{if } \varepsilon_2 \le d(x, x_0) \le \frac{\varepsilon}{2} \\ \beta & \text{if } \frac{\varepsilon}{2} \le d(x, x_0). \end{cases}$$

We shall show that

(31) $\quad g_{0|\mathbb{R}^2 \setminus \{x_0\}}$ is a continuous function.

For this purpose define $\tilde{g}_0 : [0, +\infty) \to \mathbb{R}^2$ by

$$\tilde{g}_0(y) = \begin{cases} \alpha_0 & \text{if } y = 0 \\ (\alpha_1^0 + y, \alpha_2^0 + \varphi(y)) & \text{if } y \in (0, \varepsilon_2] \\ (\alpha_1^0 + y, \alpha_2^0) & \text{if } y \in [\varepsilon_2, \frac{\varepsilon}{2}]; \\ \beta & \text{if } y \in [\frac{\varepsilon}{2}, +\infty). \end{cases}$$

Clearly

(32) $$g_0 = \tilde{g}_0 \circ d(x_0, \cdot).$$

Since φ is continuous, we may assert that $\tilde{g}_{0|(0,+\infty)}$ is continuous. By the continuity of $x \mapsto d(x_0, x)$, equality (32) implies (31). Consequently we shall show that

(33) $\quad g_0$ is a quasi-continuous function.

Since $\mathbb{R}^2 \setminus \{x_0\}$ is open, it suffices, to establish the quasi-continuity of this mapping at the point x_0. Let $\delta_1, \delta_2 > 0$. Then $\frac{\varepsilon_2}{k_0} < \min\{\delta_1, \delta_2\}$ for some $k_0 \in \mathbb{N}$. It is easy to check that $a_{k_0} = \left(\alpha_1^0 + \frac{\varepsilon_2}{k_0}, \alpha_2^0\right)$ satisfies the condition $a_{k_0} \in K(\alpha_0, \delta_2) \cap g_0(\mathbb{R}^2)$. Let $x_{k_0} \in g_0^{-1}(a_{k_0})$. Then $x_{k_0} \in K(x_0, \delta_1) \setminus \{x_0\}$. Consequently, by (31), $g_0(V) \subset K(\alpha_0, \delta_2)$ for some neighborhood V of the point x_0, such that $V \subset K(x_0, \delta_1)$ which completes the proof of (33).

Now we prove that g_0 is a Darboux function. To this end we shall first prove that

(34) $\quad \tilde{g}_0$ possesses the Darboux property.

To prove (34) it suffices to show that

(35) $\quad$ the set $\tilde{g}_0(J_{\varepsilon_0})$ is a connected set for any $\varepsilon_0 \in (0, +\infty)$,

where $J_{\varepsilon_0} = [0, \varepsilon_0]$ and $\varepsilon_0 \in (0, +\infty)$. We shall show that

$$\alpha_0 \in \overline{\tilde{g}_0(J_{\varepsilon_0} \setminus \{0\})}. \tag{36}$$

For this purpose consider a sequence $\{\xi_k\}_{k=1,2,\dots}$ where $\xi_k = \frac{\varepsilon_2}{k}$ for $k \in \mathbb{N}$. Without loss of generality assume that $\xi_k \in (J_{\varepsilon_0} \setminus \{0\}) \cap (0, \varepsilon_2)$ for $k \in \mathbb{N}$. Obviously $\varphi(\xi_k) = 0$ for $k = 1, 2, \dots$. Consequently the sequence with the general term $(\alpha_1^0 + \xi_k,\, \alpha_2^0 + \varphi(\xi_k))$ is contained in the set $\tilde{g}_0(J_{\varepsilon_0} \setminus \{0\})$ and converges to α_0, which completes the proof of (36). Since $\tilde{g}_0(J_{\varepsilon_0} \setminus \{0\})$ is connected and by the evident equality $\tilde{g}_0(J_{\varepsilon_0}) = \tilde{g}_0(J_{\varepsilon_0} \setminus \{0\}) \cup \{0\}$, (36) implies (35) which, in turn, implies (34).

Now let $C \in \mathbb{R}^2$ be an arbitrary connected set. Then the set $d(x_0, C)$ is connected and by (32), $g_0(C) = \tilde{g}_0(d(x_0, C))$. This means, by (34), that $g_0(C)$ is a connected set. The Darboux property of the function g_0 has finally been proved.

It can easily be seen that g_0 is a function with finite variation. For simplicity set $\delta = \varepsilon_1 - \frac{\varepsilon^2}{8} = \frac{\varepsilon}{2} - \frac{3}{8}\varepsilon^2$. Then $\delta \in (0, \varepsilon_1)$. Let t be an arbitrary continuous function. We shall show that $t \notin K(g_0, \delta)$. Assume to the contrary that

$$t \in K(g_0, \delta). \tag{37}$$

Set

$$A^+_{\alpha_0}(\delta) = \{(\xi, \eta) \in \mathbb{R}^2 : \eta > \alpha_2^0 - (\varepsilon_1 - \delta)\}$$

and

$$A^-_{\alpha_0}(\delta) = \{(\xi, \eta) \in \mathbb{R}^2 : \eta < \alpha_2^0 + (\varepsilon_1 - \delta)\}.$$

Obviously $A^-_{\alpha_0}(\delta) \cup A^+_{\alpha_0}(\delta) = \mathbb{R}^2$. Without loss of generality assume that $t(x_0) \in A^-_{\alpha_0}(\delta)$. Since t is continuous, we infer that

$$t(K(x_0, \delta')) \subset A^-_{\alpha_0}(\delta) \tag{38}$$

for some $\delta' \in (0, \delta)$. Let k_1 be a positive integer such that

$$\frac{2\varepsilon_2}{1 + 4k_1} < \delta' \tag{39}$$

and let x_{k_1} be a point such that $d(x_{k_1}, x_0) = \frac{2\varepsilon_2}{1+4k_1}$. Then (39) implies that $x_{k_1} \in K(x_0, \delta')$. Thus by (38), we have $t(x_{k_1}) \in A^-_{\alpha_0}(\delta)$. On the other hand (37) implies that $t(x_{k_1}) \in K(g_0(x_{k_1}), \delta)$. By the definition of g_0, we get $g_0(x_{k_1}) = \left(\alpha_1^0 + \frac{2\varepsilon_2}{1+4k_1},\, \alpha_2^0 + \varepsilon_1\right)$. Hence $t(x_{k_1}) \notin A^-_{\alpha_0}(\delta)$. This contradiction proves $t \notin K(g_0, \delta)$. It is easy to verify that $K(g_0, \delta) \subset K(f_0, \varepsilon)$. Therefore the porosity at f_0 is at least

$$2 \limsup_{\varepsilon \to 0^+} \frac{\frac{\varepsilon}{2} - \frac{3}{8}\varepsilon^2}{\varepsilon} = 1.$$

The proof of the case when f_0 is a constant function has been completed. Now proceed to the second stage of the proof of the theorem. Let $f_0 : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous with finite variation, taking at least two values, and let $\varepsilon \in (0, 3)$. Then f_0 is not constant. Hence

(40) there exists a point x_0 such that, for any neighborhood U of x_0 there is a point $z_0 \in U$ such that $f_0(x_0) \neq f_0(z_0)$.

Put $f_0(x_0) = \alpha_0 = (\alpha_0^1, \alpha_0^2)$ and $\tau = \frac{\varepsilon}{3}$. Then $\tau \in (0,1)$. Let $\tau' = \tau - \frac{\tau^2}{3} > 0$ and let $\tau_1 = \tau' - \frac{\tau^2}{3} > 0$. Moreover set $\tau_2 = \frac{\sqrt{3}}{3}\tau\sqrt{\tau(2-\tau)}$. It is not difficult to see that $\tau_1^2 + \tau_2^2 = (\tau')^2$. Let φ be the function defined by $\varphi(\xi) = \tau_1 \sin \frac{\pi\tau_2}{\xi}$ for $\xi \in (0, \tau_2]$. If $\xi \in [\tau', \tau)$, then set $k_\xi = \left[\frac{\tau - \tau'}{\tau - \xi}\right]$ where $[a]$ denotes the greatest integer $\leq a$. Let ψ be the function defined by

$$\psi(\xi) = \frac{2k_\xi(k_\xi + 1)}{\tau - \tau'}\pi\left[\xi - \frac{\tau(k_\xi - 1) + \tau'}{k_\xi}\right] \text{ for } \xi \in [\tau', \tau)$$

and let

$$h_0'(\xi) = (\alpha_1^0 + \xi\cos\psi(\xi),\ \alpha_2^0 + \xi\sin\psi(\xi)) \text{ for } \xi \in [\tau', \tau).$$

We shall prove that h_0' is a continuous mapping. Set

$$a_k = \tau - \frac{\tau - \tau'}{k} \text{ for } k = 1, 2, \ldots \tag{41}$$

and

$$J_k = [a_k,\ a_{k+1}) \text{ for } k = 1, 2, \ldots. \tag{42}$$

It can be proved that if $\xi \in J_k$, then $k = k_\xi$ for $k = 1, 2, \ldots$. By the definition of the function ψ, we conclude that

(43) for a fixed $k \in \mathbb{N}$ the function $\psi_{|J_k}$ is linear and $\psi_{|J_k}(J_k) = [0, 2\pi)$.

Let $k \in \mathbb{N}$ be fixed. Then (43) implies that the functions $\xi \mapsto \alpha_1^0 + \xi\cos\psi(\xi)$ and $\xi \mapsto \alpha_2^0 + \xi\sin\psi(\xi)$ are continuous on J_k and moreover $\lim_{\xi \to a_{k+1}^-} \cos\psi(\xi) = \cos\psi(a_{k+1})$. Thus the function $\xi \mapsto \alpha_1^0 + \xi\cos\psi(\xi)$ is continuous for $\xi \in [\tau', \tau)$.

Analogously we establish the continuity of the function $\xi \mapsto \alpha_2^0 + \xi \sin\psi(\xi)$ for $\xi \in [\tau', \tau)$. These results imply that h_0' is continuous on $[\tau', \tau)$.

Now define $\tilde{h}_0 : [0, \tau) \to \mathbb{R}^2$ by

$$\tilde{h}_0(y) = \begin{cases} \alpha_0 & \text{if } y = 0 \\ (\alpha_0^1 + y,\, \alpha_2^0 + \varphi(y)) & \text{if } y \in (0, \tau_2] \\ (\alpha_0^1 + y,\, \alpha_2^0) & \text{if } y \in [\tau_2, \tau'] \\ h_0'(y) & \text{if } y \in [\tau', \tau) \end{cases}$$

and $h_0'' : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h_0''(x) = \begin{cases} \tilde{h}_0(d(\alpha_0, x)) & \text{if } x \in K(\alpha_0, \tau) \\ x & \text{if } x \notin K(\alpha_0, \tau). \end{cases}$$

It is easy to see that

$$h''_{0|K(\alpha_0,\tau)} = \tilde{h}_0 \circ d_{|K(\alpha_0,\tau)}(\alpha_0, \cdot). \tag{44}$$

Note that

$$\tilde{h}_{0|(0,\tau)} \text{ is a continuous function.} \tag{45}$$

By (44) and (45) as well as by the definition of h_0'', we infer that

$$h''_{0|\mathbb{R}^2\setminus(S(\alpha_0,\tau)\cup\{\alpha_0\})} \text{ is a continuous function.} \tag{46}$$

Analogously to the proof of condition (34) we can show that

$$\tilde{h}_0 \text{ is a Darboux function.} \tag{47}$$

We shall now prove that

(48) if $C \subset K(\alpha_0, \tau)$ or $C \subset \mathbb{R}^2 \setminus K(\alpha_0, \tau)$, and if C is connected, then $h_0''(C)$ is also a connected set.

Indeed this fact, in the case when $C \subset \mathbb{R}^2 \setminus K(\alpha_0, \tau)$, follows immediately from (46). So assume $C \subset K(\alpha_0, \tau)$. Then $d(\alpha_0, \tau)$ is an interval contained in $[0, \tau)$. By (44) and (47), $h_0''(C) = \tilde{h}_0(d(\alpha_0, C))$ is a connected set completing the proof of (48).

Next we prove that

(49)
if C is a connected set such that $C \cap K(\alpha_0, \tau) \neq \emptyset$
and $C \setminus K(\alpha_0, \tau) \neq \emptyset$, then $\overline{h_0''(C \cap K(\alpha_0, \tau))} \supset$
$h_0''(C \cap K(\alpha_0, \tau)) \cup S(\alpha_0, \tau)$ and $h_0''(C \cap \bar{K}(\alpha_0, \tau))$ is a connected set.

Indeed by the assumptions on C, we deduce that

$$J = d(\alpha_0, C \cap K(\alpha_0, \tau)) = [a, \tau) \tag{50}$$

Hence by (44) and (47),

$$h_0''(C \cap K(\alpha_0, \tau)) \text{ is a connected set.} \tag{51}$$

Let $z \in S(\alpha_0, \tau)$. Then $z = (\alpha_1^0 + \tau \cos \psi_z,\ \alpha_2^0 + \tau \sin \psi_z)$ where $\psi_z \in [0, 2\pi)$. Note that $\lim_{k \to +\infty} a_k = \tau$ (cf. (41)). Without loss of generality assume that $a_k > a$ (cf. (50)) for each $k \in \mathbb{N}$. By (43) $\psi_z = \psi(\xi_{z,k})$ where $\xi_{z,k} \in J_k$ (cf. (42)) for $k \in \mathbb{N}$. Consequently $\xi_{z,k} \in J$ for $k \in \mathbb{N}$ and the sequence with the general term $z^k = (\alpha_1^0 + \xi_{z,k} \cos \psi_z,\ \alpha_2^0 + \xi_{z,k} \sin \psi_z)$ converges to z. By the definition of J, we infer that $\xi_{z,k} = d(\alpha_0, x_k)$ where $x_k \in C \cap K(\alpha_0, \tau)$ and $k \in \mathbb{N}$ is arbitrary. Also $\lim_{k \to +\infty} h_0''(x_k) = z$. The proof of the inclusion of (49) is complete. From this inclusion and the definition of h_0'' we can assert that

$$h_0''(C \cap K(\alpha_0, \tau)) \subset h_0''(C \cap \bar{K}(\alpha_0, \tau)) \subset \overline{h_0''(C \cap K(\alpha_0, \tau))}.$$

The above fact and (51) implies that $h_0''(C \cap \bar{K}(\alpha_0, \tau))$ is a connected set. Therefore the proof of condition (49) is finally complete.

Now let C be a connected set with $C \cap K(\alpha_0, \tau)) \neq \emptyset$ and $C \backslash K(\alpha_0, \tau) \neq \emptyset$. Let $\{\mathcal{B}_t\}_{t \in T}$ be the family of all components of $h_0''(C \backslash K(\alpha_0, \tau)) = C \backslash K(\alpha_0, \tau)$. Assume in addition that C is an arc. First establish that

$$\mathcal{B}_t \cap h_0''(C \cap \bar{K}(\alpha_0, \tau)) \neq \emptyset \text{ for } t \in T. \tag{52}$$

For simplicity set $\mathcal{H} = C \backslash K(\alpha_0, \tau)$. By the assumptions on C and by the theorem of Janiszewski, we infer that $\mathcal{B}_t \cap \mathrm{Cl}_C(C \backslash \mathcal{H}) \neq \emptyset$ where t is a fixed element of the set T. This implies $\mathcal{B}_t \cap \bar{K}(\alpha_0, \tau) \neq \emptyset$. Since $\mathcal{B}_t \cap K(\alpha_0, \tau) = \emptyset$, we have $\mathcal{B}_t \cap S(\alpha_0, \tau) \neq \emptyset$. By the last condition and the two evident inclusions $h_0''(C \cap \bar{K}(\alpha_0, \tau)) \supset C \cap S(\alpha_0, \tau)$ and $\mathcal{B}_t \subset C$ and since $t \in T$ is arbitrary, we infer (52).

Clearly $h_0''(C) = h_0''(C \cap \bar{K}(\alpha_0, \tau)) \cup \bigcup_{t \in T} \mathcal{B}_t$. The above condition as well as (51) and (52) prove that $h_0''(C)$ is connected in the case when $C \cap K(\alpha_0, \tau) \neq \emptyset$ and $C \backslash K(\alpha_0, \tau) \neq \emptyset$. This fact and (19) allows us to conclude that

$$\text{the function } h_0'' \text{ possesses the Darboux property.} \tag{53}$$

Since f_0 is continuous , we see that the image of any arc under f_0 is arcwise connected. Hence by (53), the function $h_0 = h_0'' \circ f_0$ is a Darboux transformation. Now we shall prove that h_0 is quasi-continuous. Since f_0

is continuous, (46) implies that $h_{0|\mathbb{R}^2\setminus f_0^{-1}(S(\alpha_0,\tau)\cup\{\alpha_0\})}$ is continuous. Consequently since $\mathbb{R}^2\setminus f_0^{-1}(S(\alpha_0,\tau)\cup\{\alpha_0\})$ is open, it suffices to consider the set $f_0^{-1}(S(\alpha_0,\tau)\cup\{\alpha_0\})$. We shall show that

(54) h_0 is a quasi-continuous function on the set $f_0^{-1}(\alpha_0)$.

Let $x_1 \in f_0^{-1}(\alpha_0)$ and let $\delta_1, \delta_2 > 0$. It must be shown that there exists an open set V such that

$$V \subset K(x_1,\delta_1) \text{ and } h_0(V) \subset K(\alpha_0,\delta_2). \tag{55}$$

Evidently $f_0^{-1}(S(\alpha_0,\tau))$ is closed, and $x_1 \notin f_0^{-1}(S(\alpha_0,\tau))$. Therefore δ_1 may be chosen so that

$$f_0^{-1}(S(\alpha_0,\tau)) \cap K(x_1,\delta_1) = \emptyset. \tag{56}$$

If f_0 is a constant function in some neighborhood V' of x_1, then $V = V' \cap K(x_1,\delta_1)$ satisfies (55) (since $h_0(V) = \{\alpha_0\}$). Consequently assume that there exists no neighborhood of x_1 in which the function f_0 is constant. Then there exists $x_1' \in K(x,\delta_1)$ such that $f_0(x_1') \neq f_0(x_1) = \alpha_0$. Let $d_0 = d(\alpha_0, f_0(x_1'))$. Then $d_0 > 0$. Since $d(\alpha_0, f_0(x_1')) = 0$ and since the function $x \mapsto d(\alpha_0, f_0(x))$ is continuous, we infer that

$$d(\alpha_0, f_0(x_{k_0})) = \frac{\tau_2}{k_0} \tag{57}$$

for some point $x_{k_0} \in K(x_1,\delta_1)\setminus\{x_1\}$, where $k_0 \in \mathbb{N}$ is chosen so that

$$0 < \frac{\tau_2}{k_0} < \min(d_0,\delta_2). \tag{58}$$

Since $\varphi\Big(\frac{\tau_2}{k_0}\Big) = 0$, from the definitions of h_0'' and h_0, and also from (57), we conclude that $h_0(x_{k_0}) = \Big(\alpha_1^0 + \frac{\tau_2}{k_0}, \alpha_2^0\Big)$ and as can easily be checked, $d(\alpha_0, h_0(x_{k_0})) = \frac{\tau_2}{k_0}$. Consequently by (58)

$$h_0(x_{k_0}) \in K(\alpha_0,\delta_2). \tag{59}$$

Note that $x_{k_0} \in K(x_1,\delta_1)$. Thus (56) implies that $x_{k_0} \notin f_0^{-1}(S(\alpha_0,\tau))$. From (57) and (58), we infer that $x_{k_0} \notin f_0^{-1}(\alpha_0)$. Thus (the set $\mathbb{R}^2\setminus f_0^{-1}(S(\alpha_0,\tau)\cup\{\alpha_0\})$ is open) x_{k_0} is a point of continuity of h_0 and by (59), there exists an open set V satisfying (55) proving condition (54).

Now we shall show that

(60) $\quad h_0$ is a quasi-continuous function on the set $f_0^{-1}(S(\alpha_0,\tau))$.

Let $x \in f_0^{-1}(S(\alpha_0,\tau))$ and let $\delta_1,\delta_2>0$. It must be shown that $h_0(V)\subset K(h_0(x),\delta_2)$ for some open set $V\subset K(x,\delta_1)$. With no loss of generality assume that

$$\delta_2 < \tau - \tau'. \tag{61}$$

By the definition of the function f_0, we infer that

$$f_0(K(x,\delta_1')) \subset K(f_0(x),\delta_2) = K(h_0(x),\delta_2) \tag{62}$$

for some $\delta_1' \in (0,\delta_1)$. Note that if there is a non-empty open set $V\subset K(x,\delta_1')$ such that $f_0(V)\subset \mathbb{R}^2\backslash K(\alpha_0,\tau)$, then, by (62), $h_0(V)\subset K(h_0(x),\delta_2)$. So now assume that the image (under f_0) of each non-empty, open subset of the ball $K(x,\delta_1')$ has a non-empty intersection with $K(\alpha_0,\tau)$. In particular by (62), $f_0(z)\in K(\alpha_0,\tau)\cap K(f_0(x),\delta_2)$ for some $z\in K(x,\delta_1')$. The set $f_0(K(x,\delta_1'))$ is connected and contains $f_0(x)$ and $f_0(z)$. Since $d_0' = d(\alpha_0,f_0(z)) < \tau$ and since $d(\alpha_0,f_0(x))=\tau$, by the continuity of the function $\xi\mapsto d(\alpha_0,f_0(\xi))$,

(63) $\quad$ for $d\in(d_0',\tau)$, there is $x_d\in K(x,\delta_1')$ with $d(\alpha_0,f_0(x_d))=d$.

By condition (49) (for $C=\mathbb{R}^2$) and since $h_0(x)=f_0(x)\in S(\alpha_0,\tau)$, we conclude that $h_0(x)\in\overline{h_0''(K(\alpha_0,\tau))}$. Thus for some y we have

$$y\in h_0''(K(\alpha_0,\tau))\cap K(h_0(x),\tau-d_0'). \tag{64}$$

So $d(\alpha_0,y)\in(d_0',\tau)$. By (63), $d(\alpha_0,f_0(\hat{x}))=d(\alpha_0,y)$ for some $\hat{x}\in K(x,\delta_1')$. By (61) $d(\alpha_0,f_0(\hat{x}))\in(\tau',\tau)$. Thus the definition of h_0'' implies $h_0''(f_0(\hat{x}))\in(\tau',\tau)$. using (64) it follows easily that $y\in h_0'((\tau',\tau))$ and consequently $y=h_0'(d(\alpha_0,y))=h_0''(f_0(\hat{x}))$. By (61) and (64), $y=h_0(\hat{x})\in K(h_0(x),\delta_2)$. Since $f_0(\hat{x})\notin S(\alpha_0,\tau)\cup\{\alpha_0\}$, the element $\hat{x}\in K(x_1,\delta_1')$ is a point of continuity of h_0, which means that there exists an open set $V\subset K(x_1,\delta_1')$ such that $h_0(V)\subset K(h_0(x),\delta_2)$. This proves (60).

It is easy to see that

$$\mathcal{N}_{h_0|\mathbb{R}^2\backslash(K(\alpha_0,z)\cap h_0(\mathbb{R}^2))}(y)=\begin{cases}0 & \text{if } y\in K(\alpha_0,\tau)\backslash h_0(\mathbb{R}^2)\\ \mathcal{N}_{f_0}(y) & \text{if } y\in\mathbb{R}^2\backslash K(\alpha_0,\tau).\end{cases}$$

Since $m_2(K(\alpha_0,\tau)\cap h_0(\mathbb{R}^2))=0$ and since f_0 has finite variation, we infer that h_0 also has finite variation. For simplicity set $\hat{\delta}=\tau_1-\dfrac{\tau^2}{6}=\tau-\dfrac{5}{6}\tau^2>0$. Then $\hat{\delta}\in(0,\tau_1)$. We shall show that

(65) $\quad t\notin K(h_0,\hat{\delta})$ for any continuous function $t:\mathbb{R}^2\to\mathbb{R}^2$.

Assume to the contrary that, for some continuous function t, the relation

$$t \in K(h_0, \hat{\delta}) \tag{66}$$

holds. Set

$$B^+_{\alpha_0}(\hat{\delta}) = \{(\xi, \eta) \in \mathbb{R}^2 : \eta > \alpha_2^0 - (\tau_1 - \hat{\delta})\}$$

and

$$B^-_{\alpha_0}(\hat{\delta}) = \{(\xi, \eta) \in \mathbb{R}^2 : \eta < \alpha_2^0 + (\tau_1 - \hat{\delta})\}.$$

Evidently, $B^+_{\alpha_0}(\hat{\delta}) \cup B^-_{\alpha_0}(\hat{\delta}) = \mathbb{R}^2$. Without loss of generality assume that $t(x_0) \in B^-_{\alpha_0}(\hat{\delta})$. Since t is continuous,

$$t(K(x_0, \hat{\delta}')) \subset B^-_{\alpha_0}(\hat{\delta}) \tag{67}$$

for some $\hat{\delta}' > 0$. Then by (40), there exists $z_0 \in K(x_0, \hat{\delta}')$ such that $f_0(x_0) \neq f_0(z_0)$. Note that the set $\mathcal{T} = f_0([x_0, z_0])$ is connected and $f_0(x_0)$, $f_0(z_0) \in \mathcal{T}$. Choose $k_1 \in \mathbb{N}$ such that

$$\frac{2\tau_2}{1 + 4k_1} < d(\alpha_0, f_0(z_0)). \tag{68}$$

The function $y \mapsto d(\alpha_0, y)$ is continuous. Since $\alpha_0 = f_0(x_0)$ since $f_0(z_0) \in \mathcal{T}$ since $\mathcal{T}$ is connected and from (68), we conclude that $d(\alpha_0, y_{k_1}) = \dfrac{2\tau_2}{1 + 4k_1}$ for some $y_{k_1} \in \mathcal{T}$. The definition of $\mathcal{T}$ implies

$$f_0(x_{k_1}) = y_{k_1} \text{ for some } x_{k_1} \in [x_0, z_0] \subset K(x_0, \hat{\delta}'). \tag{69}$$

In turn the definition of h_0 implies $h_0(x_{k_1}) = \left(\alpha_1^0 + \dfrac{2\tau_2}{1 + 4k_1}, \alpha_2^0 + \tau_1\right)$ and by (66), we infer that $t(x_{k_1}) \in K(h_0(x_{k_1}), \hat{\delta})$. Therefore $t(x_{k_1}) \notin B^-_{\alpha_0}(\hat{\delta})$. However from (67) and (69) we easily conclude that $t(x_{k_1}) \in B^-_{\alpha_0}(\hat{\delta})$. This contradiction proves that (66) is false thereby completing the proof of (65).

It can easily be verified that $\rho(h_0, f_0) \leq 2\tau$. Consequently the ball $K(f_0, 3\tau)$ contains a ball $K(h_0, \hat{\delta})$ such that $K(h_0, \hat{\delta})$ does not contain any continuous function. Thus the porosity at the point f_0 is not less than the number

$$2 \cdot \limsup_{\varepsilon \to 0^+} \frac{\frac{\varepsilon}{3} - 5 \cdot \frac{\left(\frac{\varepsilon}{3}\right)^2}{6}}{\varepsilon} = \frac{2}{3}.$$

$\square$

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