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TWO-DIMENSIONAL PARTITIONS

1. Introduction

For the purposes of this article,

A *rectangle* means a Cartesian product in the plane, $\mathbb{R}^2$, of two compact intervals, $R = [a, b] \times [c, d]$.

A *partition* of a rectangle R, is a finite collection of nonoverlapping subrectangles whose union is R.

For $0 < \lambda \leq 1$ a partition is called $\lambda - regular$ if each subrectangle satisfies $\lambda \leq eccentricity = \min\{height/width, width/height\}$.

A *gauge* is any function from the plane to the positive real numbers.

A partition is called *special* $\delta - fine$ if every subrectangle has at least one vertex, where the gauge, δ, is larger than both the height and the width of the rectangle. (The word "special" denotes that the gauge is calculated at a vertex of the rectangle, rather than at just any point in the rectangle.)

Given particular δ and λ, we call a partition *proper* if it is both λ-regular and also special δ-fine.

In [1] Z. Buczolich, answering a question of W. Pfeffer (see [3]) established the following assertion.

Theorem 1.1 (Buczolich) *If* $\lambda = \frac{1}{1000}$ *and if* δ *is any gauge, then every rectangle can be properly partitioned.*

No attempt was made by Buczolich to improve the *regularity* λ, since for his purposes any $\lambda > 0$ was sufficient. In fact, the λ in his proof is actually bigger than 1/1000. As it turned out the Riemman type integral which follows from this theorem did not have the properties Pfeffer was hoping for and so the study of special partitions seemed to be abandoned. (See [4].)

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Our interest in Buczolich's theorem is from a different point of view. We would like to know how much λ could be improved and in particular, could λ be arbitrarily close to 1? We think this question is natural and intriguing from a geometrical point of view. Furthermore if the regularity λ could be shown to be bounded below by 1, then the same bound may suffice for similar problems involving "symmetrical δ-fine" partitions (where the center of the rectangle, rather than vertices is used), and this may have important ramifications in the theory of uniqueness for trigonometric series.

We therefore did our best to try to improve the λ in Buczolich's theorem. We found that λ could be any number $< \frac{1}{\sqrt{2}}$. In the case where δ is upper semicontinuous, then λ could be exactly $\frac{1}{\sqrt{2}}$. We conjecture that this is the best possible. The reason (as will become apparent in the following proof) is that when two rectangles in the construction are found to overlap, there seems to be an overwhelming need to extend the short side of the larger rectangle to encompass the smaller one. Hence we need to have a rectangle such that both ratios *height/width* and $2 \cdot$ *height/width* are between λ and $1/\lambda$ and that is why $\lambda = \frac{1}{\sqrt{2}}$ is used. Although this value of λ seems natural, we have no idea how to prove it is best possible.

Our main result is as follows.

Theorem 1.2 *If $\lambda < \frac{1}{\sqrt{2}}$ and if δ is any gauge, then every rectangle can be properly partitioned.*

2. Pink Sets

Let δ be a positive function defined on $\mathbb{R}^2$ and let $0 < \lambda < 1$. We say that an open set G is *pink* if every rectangle $A \subset G$ has a proper partition. First we show that the union of all pink sets is a dense subset of $\mathbb{R}^2$, and is also pink.

To do this, let $A_n = \{(x, y) \,|\delta(x, y) \geq 1/n\}$ and let $S = \cup_{n=1}^{\infty}(\overline{A_n})^0$ where F^0 denotes the *interior* of a set F. By the definition of δ, $\cup_{n=1}^{\infty} A_n = \mathbb{R}^2$. Hence by the Baire Category Theorem, S is nonempty. To show that S is pink, let $B \subset S$ be a rectangle. Since B is compact, it is contained in the union of finitely many $\overline{A_n}^0$. But the $\overline{A_n}$ are nested; so there is a single $\overline{A_n}$ such that $B \subset \overline{A_n}^0$. Without loss of generality suppose that the diameter of B is smaller than 1/n and that B has eccentricity bigger than λ. If the center, c, of B is from A_n, cut B into four rectangles with a common vertex c. If $c \in \overline{A_n} \setminus A_n$, then choose $d \in A_n$ close enough to c so that the four rectangles that partition B with a common vertex d still have eccentricity bigger than λ. In either case the four rectangles are a δ-fine partition of B. Therefore according to our definition, the open set S is pink. Repeating the argument in any closed interval shows that the pink sets are dense.

Let $P = \cup\{G \mid G \text{ is pink}\}$. If P were not pink, then there would be a rectangle $B \subset P$ which could not be properly partitioned. Quadrasecting using the center of B yields four smaller rectangles at least one of which can't be properly partitioned. Continuing this process we obtain a sequence of rectangles which cannot be properly partitioned. The sequence converges to a point and hence eventually the tail of the sequence will be inside the open set G that contains that point. This contradicts the fact that G is pink. Therefore P is pink.

Let $C = \mathbb{R}^2 \setminus P$. Suppose $C \neq \emptyset$. By the Baire Category Theorem there is a rectangle D and an integer n so that $\emptyset \neq C \cap D^0 \subset \overline{A_n}$ and $diam(D) < 1/n$. We fix such an n. Notice that by the definition of P there is a rectangle $R \subset D$ that can't be properly partitioned. For $\lambda < \frac{1}{\sqrt{2}}$ we will show how to properly partition R contradicting that $C \neq \emptyset$. This means that all of $\mathbb{R}^2$ is pink which will finish the proof. For the rest of the article $0 < \lambda < \frac{1}{\sqrt{2}}$.

It remains to show that every rectangle $R \subset D$ can be properly partitioned. So, let $R = [a,b] \times [c,d] \subset D$ be given. We show that if $\epsilon(x) = \min\{\delta(a,x), \delta(b,x), (b-a)/2\}$ and if $c \leq x \leq d$, then for every $0 < \epsilon \leq \epsilon(x)$ the rectangle $[a,b] \times [x, x+\epsilon]$ can be properly partitioned. It is easy to see that the one dimensional interval $[c,d]$ can be partitioned into a finite sequence of nonoverlapping intervals, I_k $k = 1, 2, \cdots, m$, of the form $I_k = [x, x+\epsilon]$ where $0 < \epsilon \leq \epsilon(x)$. This fact is known in the literature as Cousins Lemma. (See [2].) Now let Ψ_k denote the collection of nonoverlapping rectangles that properly partition $[a,b] \times I_k$. Then the finite collection of $\{\Psi_k\}_{k=1}^m$ is a proper partition of R. So to complete the proof it remains to show that $[a,b] \times I_k$ can be properly partitioned. Fix k and let Y denote $[a,b] \times I_k$.

Remark It is easy to see that every rectangle with pink interior can be properly partitioned. To do this, first put a δ-fine square in each corner. Then, using Cousins lemma, partition each side with δ-fine squares. Then partition the remainder into pink rectangles which then can be properly partitioned.

3. Green Points and Thin Rectangles

Definition 3.1 *We say that a rectangle B is thin if*

a) At least one left vertex and at least one right vertex have a gauge value $\delta > \sqrt{2} \cdot height(B)$.

b) $width(B) > \sqrt{2} \cdot height(B)$.

Notice that $Y = [a,b] \times I_k$ is thin. Call the points from $Y \cap A_n$ *green*. Let $\operatorname{proj}(A)$ denote the projection of the set A onto the interval $[a,b]$. Let $h(x)$

denote the distance from x to the base of Y. If Υ is a collection of sets, let $T_\Upsilon = T_\Upsilon = ([a,b] \setminus \operatorname{proj}(\cup_{A \in \Upsilon} A)) \times I_k$.

Using $H = height(Y)$, construct two rectangles in Y at each end of Y with height H and with width $H/\sqrt{2}$. Since Y is thin, the two rectangles are $\frac{1}{\sqrt{2}}$-regular, δ-fine and nonoverlapping. Let Ψ denote the collection of these two rectangles. We establish the following algorithm to inductively define subpartitions of Y. After showing this procedure terminates in finitely many steps, we can produce the desired partition of Y. Before we introduce the algorithm, for a point $x \in Y$, $R_l = R_l(x)$ and $R_r = R_r(x)$ will be two rectangles with:

a) common vertex x,

b) bases on the base of Y.

See Fig 1.

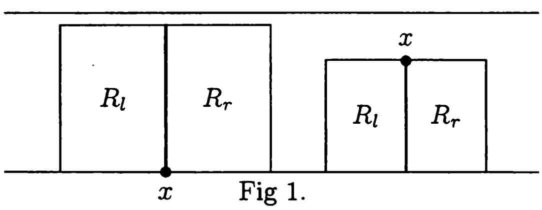

Fig 1.

Algorithm Step 0. Find $x \in \overline{T_\Psi}$ on the top side of Y such that $\delta(x) \geq H\sqrt{2}$, if possible and construct rectangles $R_l(x)$ and $R_r(x)$ of dimensions $\frac{H}{\sqrt{2}} \times H$. Otherwise go to Step 2.
Step 1. Adjoin to Ψ the rectangles $R_l(x)$ and $R_r(x)$ and go back to Step 0.
Step 2. Find $x \in \overline{T_\Psi}$ on the bottom side of Y such that $\delta(x) \geq H\sqrt{2}$, if possible and construct rectangles $R_l(x)$ and $R_r(x)$ of dimensions $\lambda H \times H$. Otherwise go to Step 4.
Step 3. Adjoin to Ψ the rectangles $R_l(x)$ and $R_r(x)$ and go back to Step 2.
Step 4. Set $h = \sup\{h(x) : x \text{ in } T_\Psi \text{ is green}\}$. Note that $h \leq H$. Let

$$S = \{x \in \overline{T_\Psi} : \exists \text{ a sequence } \{x_m\} \subset T_\Psi \cap A_n \text{ converging to } x \text{ and } h(x) = h\}.$$

If $S = \emptyset$, go to Step 5. Otherwise for each $x \in S$ construct rectangles R_l and R_r of dimensions $\frac{h}{\sqrt{2}} \times h$. Since $\sup_{x,y \in S} |x - y|$ is finite, we may pick a finite subset, $\{x_1, \ldots, x_m\}$, of S so that, for $1 \leq i < j \leq m$, $|x_i - x_j| \geq \frac{h}{\sqrt{2}}$ and $S \subset \cup_i (R_l(x_i) \cup R_r(x_i))$. Renumber the sequence, if necessary, from left to right. Note that by the construction there is $0 < t < \frac{h}{2}$ such that

$$([a,b] \setminus \operatorname{proj}(\cup_{R \in \Psi} R \cup_i (R_l(x_i) \cup R_r(x_i)))) \times [h-t, h]$$

is pink.

Let W be a component of $\cup_i(R_l(x_i) \cup R_r(x_i))$, u, v the upper vertices of W. See Fig 2.

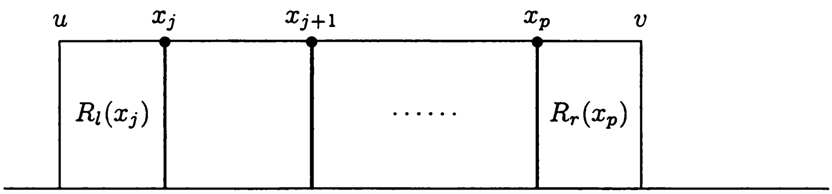

Fig 2.

Observe that W can only intersect $\cup_\Psi R$ in $R_l(x_j)$ or $R_r(x_p)$. Successively pick points $y_i \in T \cap A_n$ $i = j, \ldots, p$ such that

i) $|x_i - y_i| < t$ and if $h < height(Y)$, then also $|x_i - y_i| < height(Y) - h$,

ii) the rectangle W_j, with bottom side $[\text{proj}\,(u), \text{proj}\,(y_j)]$ and with upper right vertex y_j has eccentricity strictly between λ and $\frac{1}{2\lambda}$.

iii) for $j < i \leq p$ the rectangle, W_i, with bottom side $[\text{proj}\,(y_{i-1}), \text{proj}\,(y_i)]$ and with upper right vertex y_i is λ-regular

iv) the rectangle W_{p+1}, with bottom side $[\text{proj}\,(y_p), \text{proj}\,(v)]$ and with upper left vertex y_p has eccentricity strictly between λ and $\frac{1}{2\lambda}$.

v) The rectangles (dashed rectangles in Fig 3.) sitting on top of rectangles W_i, and with top sides on the top side of W are thin. See Fig 3.

Observe that ii) and iv) are possible since $\lambda < \frac{1}{\sqrt{2}} < \frac{1}{2\lambda}$. Statement iii) is possible since $\frac{h}{\sqrt{2}} \leq |x_i - x_{i-1}| \leq h\sqrt{2}$ and v) is possible since it suffices to pick y_i close enough to x_i such that the height of the dashed rectangle is smaller than the gauge of the vertex opposite to y_i.

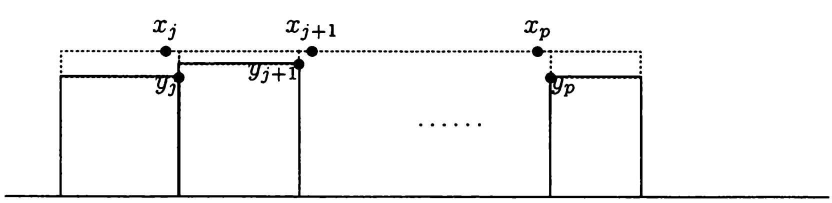

Fig 3.

Note that if there is a $P \in \Psi$ such that $P^0 \cap W \neq \emptyset$, then either $P^0 \cap W_j^0 \neq \emptyset$ or $P^0 \cap W_{p+1}^0 \neq \emptyset$, and $height(P)$ is greater than h, and the eccentricity of

P is strictly between λ and $\frac{1}{2\lambda}$. Adjoin the rectangles $W_j, \ldots, W_{p+1}$ to Ψ, set H =the minimum height of rectangles in Ψ, and go to Step 2.

Step 5. If none of the previous steps can be performed, the algorithm is finished.

First we have to show that the algorithm is finite. Suppose not. If H stays bounded away from 0, then the widths of rectangles in Ψ are also bounded away from 0. Since intersection of every three rectangles has empty interior, this is impossible. Since H is nonincreasing, H converges to 0. In this case there is a sequence of green corners converging to a point x on the base of Y which is in $\overline{Y \setminus \cup_{R\in\Psi} R}$. This forces Step 2 in the algorithm to be applied every time after some stage. However when Step 2 is applied the value of H does not decrease, contradicting that $H \to 0$. Therefore the algorithm is finite.

Some rectangles in the collection Ψ may overlap. But no three rectangles have common overlap, and if two of them do overlap, say $P = J \times I$ and $Q = E \times F$, then both have eccentricities strictly between λ and $\frac{1}{2\lambda}$. So if P is taller than Q (in which case P was constructed in a stage before Q), simply replace these two rectangles by $(J \cup E) \times I$. The new collection is nonoverlapping λ-regular and δ-fine.

Therefore we partitioned Y into a finite collection of nonoverlapping rectangles such that they are either λ-regular and δ-fine or with pink interior or thin.

The only ones that we still need to consider are those that are thin. See Fig 4. below. (Dashed rectangles are the thin rectangles.) If the top side of a thin rectangle, R, is not on the top side of Y, then the height of the pink region in Y above R is, by i), greater than the height of R. Apply the algorithm to R and let R_1 be a thin rectangle in the partition of R. Since R_1 has a green lower vertex, x, we can replace R_1 with a rectangle of the same width as R_1 and height $h(x)$ (the distance from x to the base of R). This rectangle is λ-regular and δ-fine and is completely inside Y even though it could extend above R.

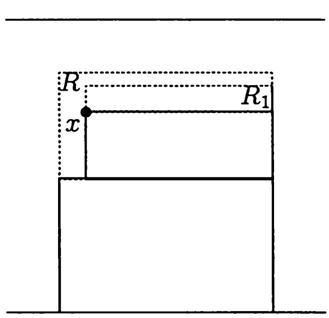

Fig 4.

So the remaining case is that of repeated application of the algorithm to a nested sequence of thin rectangles $\{R_k\}$, all having top side on the top side of Y. Let $x \in \cap_k R_k$. Note that x is on top side of every R_k. Then there is a k such that x is a green point. Since x is a green point on the top side of R_k, there is a δ-fine $\frac{1}{\sqrt{2}}$ regular rectangle in the partition of R_k that contains x which is a contradiction. This completes the proof of the Theorem.

4. Upper Semicontinuous Case

Finally note that if δ is upper semicontinuous, then A_n is closed. So for the points y_i in Step 4 one can choose $y_i = x_i$, in which case for $j < i \leq p$ the rectangles W_i are $\frac{1}{\sqrt{2}}$-regular, while the eccentricities of W_j and W_{p+1} are exactly $\frac{1}{\sqrt{2}}$. So if $\lambda = \frac{1}{\sqrt{2}}$, then if P and Q are two rectangles in Ψ that overlap, then their union is still $\frac{1}{\sqrt{2}}$-regular. Since also in this case the set of thin rectangles from step 4 is empty we have the following theorem:

Theorem 4.1 *If δ is a positive upper semicontinuous function defined on $\mathbb{R}^2$, then every rectangle has a $\frac{1}{\sqrt{2}}$-regular δ-fine partition.*

References

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