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BÔCHER'S THEOREM IN $\mathbb{R}^2$ AND CARATHÉODORY'S INEQUALITY

Abstract

An improved version of the classical Bôcher's theorem in $\mathbb{C}$ is proved using Carathéodory's inequality and Riesz integral representation of subharmonic functions.

1. Introduction

Bôcher's theorem (see Axler et al [1]) in the complex plane states that if $u(z) \geq 0$ is harmonic in $0 < |z| < 1$, then $u(z) = \alpha \log |z| + v(z)$, where $v(z)$ is harmonic in $|z| < 1$ and $\alpha \leq 0$.

In this note, we show that this theorem can be proved with only a one-sided asymptotic growth condition on u. The tools used for this new proof are the Carathéodory's inequality and the Riesz integral representation for subharmonic functions. This form of Bôcher's theorem is sharper than the ones given in [1].

2. Bôcher's Theorem in $\mathbb{C}$

Lemma 1 *Let $h(z)$ be harmonic in $|z| > R$. Let $a > R$. Then, in $|z| > a$, $h(z) = \int \log |z - \xi| \, d\mu(\xi) + H(z)$, where $H(z)$ is harmonic in $\mathbb{C}$ and μ is a signed measure with compact support in $K = \{z : |z| = a\} \cup \{0\}$.*

PROOF. Given a continuous function f on $|z| = a$, let $D_a f$ denote the continuous function in $|z| \leq a$ such that $D_a f$ is harmonic in $|z| < a$ and $D_a f = f$ on $|z| = a$ (See Poisson Integral formula, R. P. Boas [2], page 167). Choose $\alpha > 0$ large, so that $h(z) - \alpha \log \frac{|z|}{a} \geq D_a h(z)$ on $|z| = b$, $R <$

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$b < a$. This is always possible since $\log\frac{|z|}{a} < 0$ in $|z| < a$. Since, on $|z| = a$, $D_a h(z) = h(z) = h(z) - \alpha\log\frac{|z|}{a}$, by maximum principle for harmonic functions, $h(z) - \alpha\log\frac{|z|}{a} \geq D_a h(z)$ in $b \leq |z| \leq a$. Define

$$u(z) = \begin{cases} h(z) & \text{in} \quad |z| > a \\ D_a h(z) + \alpha\log\frac{|z|}{a} & \text{in} \quad |z| \leq a \end{cases}$$

Then $u(z)$ is harmonic outside $K = \{z : |z| = a\} \cup \{0\}$, $u(z)$ is subharmonic in $|z| < a$ and $-u(z)$ is subharmonic in $|z| > 0$. Consequently, by Riesz decomposition theorem for subharmonic functions (M. Tsuji [3], p. 48), if $0 < b < a < c$, there exists a unique (positive) measure μ_1 with support in $|z| = a$ such that if $b < |z| < c$, then $-u(z) = v(z) + \int \log|z - \xi|\, d\mu_1(\xi)$ where $v(z)$ is harmonic in $b < |z| < c$.

Note that $\int \log|z-\xi|\, d\mu_1(\xi)$ is subharmonic in $\mathbb{C}$ and harmonic outside the circle $|z| = a$. Also observe that if δ is the Dirac measure with support $\{0\}$, then $u(z) = D_a h(z) - \alpha\log|a| + \alpha\int\log|z-\xi|\, d\delta(\xi)$ for $|z| < b$ where $D_a h(z) - \alpha\log|a|$ is harmonic in $|z| < a$. Set $\mu = \alpha\delta - \mu_1$. Then μ is a signed measure with support in $K = \{z : |z| = a\} \cup \{0\}$ such that in $\mathbb{C}$, $\int\log|z-\xi|\, d\mu(\xi)$ is well-defined and $H(z) = u(z) - \int\log|z-\xi| d\mu(\xi)$ is harmonic in $\mathbb{C}$. Since $u(z) = h(z)$ in $|z| > a$, the lemma is proved. □

Note: In the above construction, set $\mu(\mathbb{C}) = \int_{\mathbb{C}} d\mu = \alpha - \mu_1(z : |z| = a)$. Then $b(z) = \int\log|z-\xi|\, d\mu(\xi) - \mu(\mathbb{C})\log|z|$ is harmonic in $|z| > a$; further, since $b(z)$ can be written as $\int_K(\log|z-\xi| - \log|z|)\, d\mu(\xi)$ where K is the compact support of μ and since $(\log|z-\xi| - \log|z|) \to 0$ as $|z| \to \infty$ and $\xi \in K$, we note that $b(z)$ is a harmonic function in $|z| > a$ tending to 0 at infinity.

Lemma 2 *Let $\varphi(z) = o(|z|^s)$ when $|z| \to \infty$ be a real-valued function with $s \leq 1$. Suppose $H(z)$ is a harmonic function in $\mathbb{C}$ such that $H(z) \geq \varphi(z)$ outside a compact set. Then $H(z)$ is a constant.*

PROOF. If $s \leq 0$, $H(z)$ has a lower bounded outside a compact set and hence $H(z)$ is a constant. (This is a version of Liouville's theorem for harmonic functions).

When $0 < s \leq 1$, choose an entire function $f(z)$ in $\mathbb{C}$ such that Re $f(z) = -H(z)$. Let $A(r) = \max_{|z|=r} -H(z)$. Note that, by hypothesis, for $\varepsilon > 0$ there exists ℓ such that, $-H(z) \leq -\varphi(z) < \varepsilon|z|^s$ for $|z| > \ell$ and hence $A(r) < \varepsilon r^s$ if $r > \ell$. Then by Carathéodory's inequality (R. P. Boas [2], page 135),

$$|f(re^{i\theta})| \leq |f(0)| + \frac{2r}{R-r}[A(R) + H(0)], \quad 0 < r < R.$$

Taking $R = 2r$, we note that $|f(z)| = o(|z|^s)$ when $|z| \to \infty$. This implies that $f(z)$, and consequently $H(z)$, is a constant. □

Theorem 1 *Let $\varphi(z) = o(|z|^{-s})$ when $|z| \to 0$ be a real-valued function with $s \leq 1$. Suppose $u(z)$ is harmonic in $0 < |z| < 1$ such that $u(z) \geq \varphi(z)$. Then $u(z) = \lambda \log|z| + v(z)$ where $v(z)$ is harmonic in $|z| < 1$.*

PROOF. Let $h(z) = u(\frac{1}{\bar{z}})$. Then $h(z)$ is harmonic in $|z| > 1$ and $h(z) \geq \varphi(\frac{1}{\bar{z}}) = o(|z|^s)$ when $|z| \to \infty$. Now, by Lemma 1, $h(z) = \int \log|z - \xi|\, d\mu(\xi) + H(z)$ in $|z| > a > 1$, where $H(z)$ is harmonic in $\mathbb{C}$. Hence, in $|z| > a$, $h(z)$ is of the form $h(z) = \mu(\mathbb{C}) \log|z| + b(z) + H(z)$ where $b(z)$ is a harmonic function in $|z| > a$, tending to 0 at infinity. (See the Note following Lemma 1). Consequently, outside a compact set, $H(z) \geq \varphi(\frac{1}{\bar{z}}) - \mu(\mathbb{C}) \log|z| - b(z)$. Let $\varphi_1(z) = \varphi(\frac{1}{\bar{z}}) - \mu(\mathbb{C}) \log|z| - b(z)$. Now, when $0 < s \leq 1$, $\varphi_1(z) = o(|z|^s)$ since $\varphi(\frac{1}{\bar{z}}) = o(|z|^s)$ and $\log|z| = o(|z|^s)$.

When $s \leq 0$, if we choose any δ such that $0 < \delta \leq 1$, then $\varphi_1(z) = o(|z|^\delta)$ since $\varphi(\frac{1}{\bar{z}}) = o(|z|^s) = o(|z|^\delta)$ and $\log|z| = o(|z|^\delta)$. Thus, for any value of $s \leq 1$, $\varphi_1(z) = o(|z|^\alpha)$ for some α, $0 < \alpha \leq 1$ and $H(z) \geq \varphi_1(z)$ outside a compact set. Hence, by Lemma 2, $H(z)$ is a constant. Thus, $h(z) = \mu(\mathbb{C}) \log|z| + b(z) +$ a constant and hence in $0 < |z| < 1$, $u(z) = h(\frac{1}{\bar{z}}) = -\mu(\mathbb{C}) \log|z| +$ (a harmonic function $v(z)$ bounded near 0). Since a bounded harmonic function in $0 < |z| < t$ extends as a harmonic function in $|z| < t$, $v(z)$ is harmonic in $|z| < 1$. (To prove this, one makes use of the Poisson integral formula and the maximum principle for harmonic functions). Thus $u(z) = \lambda \log|z| + v(z)$, where $\lambda = -\mu(\mathbb{C})$. □

Remark: Now it is important to remark that this theorem gives the best possible result. The example of the function $u(z) = \mathrm{Re}\frac{z}{|z|^2}$ which is harmonic in $0 < |z| < 1$ and $u(z) = O(|z|^{-1})$ shows that in the theorem $o(|z|^{-s})$ cannot even be replaced by $O(|z|^{-s})$. This form of Bôcher's theorem is sharper than the ones given in Axler et al. ([1], Theorem 3.9 and Exercise 16, p. 58).

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